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CABLE DYNAMICS

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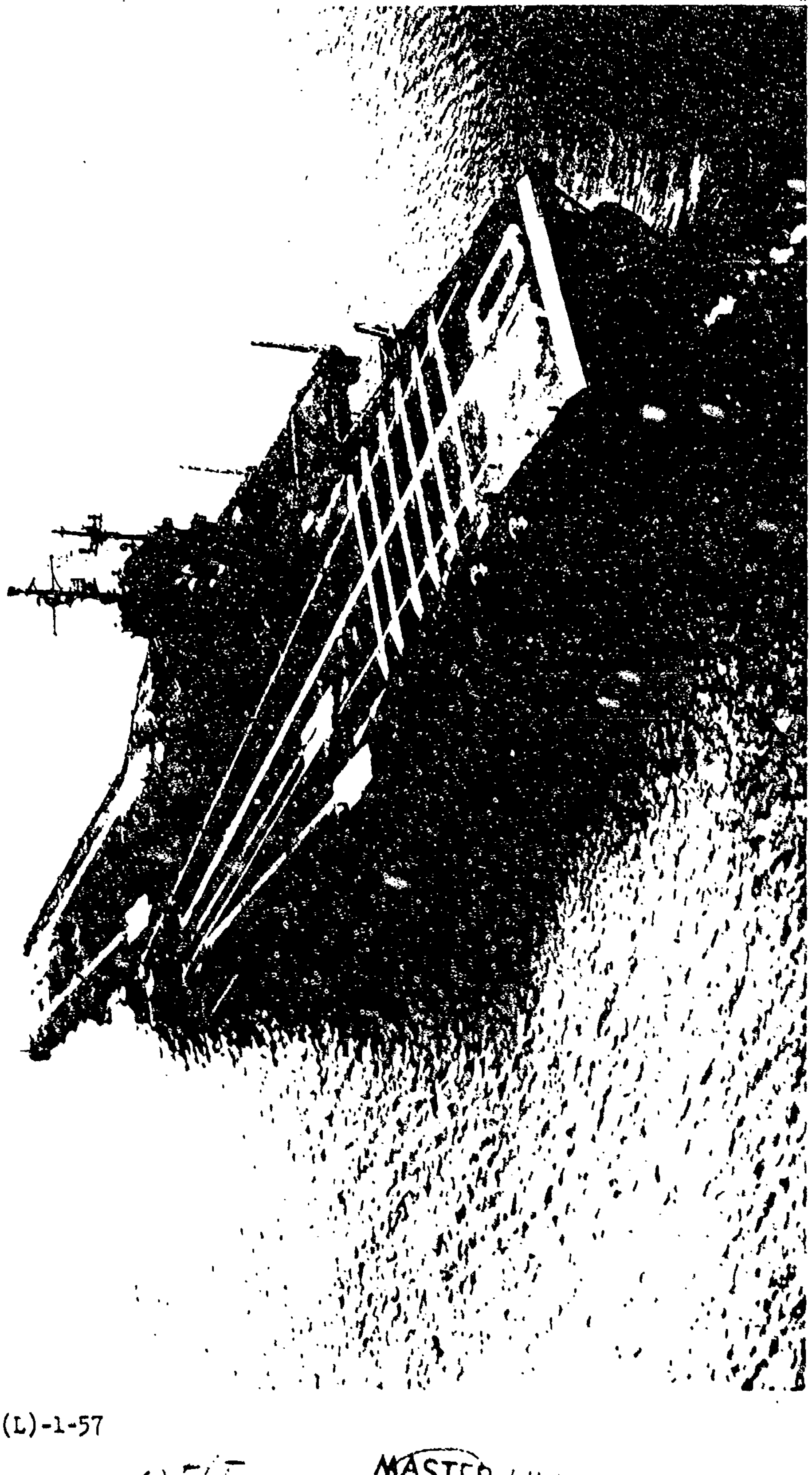

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CABLE DYNAMICS

By Friedrich O. Ringleb

PREFACE

The technical development of the aircraft arresting gear required the solution of a considerable number of problems on cable dynamics which had not previously been studied. Accordingly, during the last ten years, this development was accompanied by extensive basic investigations on cable dynamics both in this country and abroad. The results of these studies, though initiated by the aricraft arresting gear development, are of general technical and scientific interest and in their application not restricted solely to this device. The purpose of the present monograph is to furnish a systematical treatment of cable dynamics, with particular respect to recent developments in, but without restriction to, aircraft arresting gear application. The examples which are included refer generally to arresting devices, and the numerous comparisons between theoretical and measured results make use of much of the experimental work conducted during arresting gear development tests.

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INTRODUCTION

Under cable dynamics we understand most generally the relations which describe the motion of an elastic cable and the stress distribution in it if an initial motion, an initial stress distribution and the forces are prescribed which act on the cable from outside during its motion. The initial motion, say at the time zero, is given by an arbitrarily prescribed position of each cable point and an arbitrarily prescribed velocity and direction of motion of each cable point at this time. The initial stress distribution is given by an arbitrarily prescribed stress in each cable point at this time. The outside forces acting on the cable during its motion are given by its magnitude and direction in each cable point as functions of their initial positions and the time.

In praxis, the initial state of motion will mostly be the state of rest, and the initial stress will be a constant stress from which the motion and a transient stress distribution will develop due to the action of outside forces.

The transmission of outside forces to a cable involves fixed or moving masses attached to the cable or moving along the cable like anchors, links, hooks, sheaves and pulleys. It is the interaction of such masses with the cable motion which yields the more complicated problems of cable dynamics.

Among the forces acting from outside on the cable, the impact forces are of special interest. Such forces are typical for the aircraft arresting gear. Here a stretched cable is engaged suddenly by the tailhook of the landing airplane with its landing speed in a direction perpendicular to the cable. The force acting on the cable rises in this case suddenly from zero to a finite value producing an infinite acceleration at an infinitely small mass of the cable. The solution of such and similar impact problems comprises a main part of more recent developments in cable dynamics.

In the following investigations, the elastic cable which is inhomogenous at least in structure and mostly also in the material will be replaced by an equivalent elastic homogenous string in a way to be specified later. Elastic strings have been studied for more than a hundred years. The classical theory of the elastic string contained in most textbooks* on mechanics deals with the motion of such string under the simplifying assumption that the stress within the string is approximately constant and that the displacements of its elements and the slopes of the string remain small during its motion. These assumptions of the linearized theory are too narrow for modern applications, especially for an application to the aircraft arresting gear where none of these assumptions is satisfied.

More general in scope would be the method in which the cable is replaced by a chain of small masses and massless springs. However, for a large number of such masses and springs, this method becomes hopelessly complicated, and approximations with small numbers of such masses and springs yield results which are too inaccurate to be of value.

A first general non-linear theory of the vibrating string in analogy to the Fourier-analysis in the classical case has been worked out in 1945 by G. F. Carrier (1) who determined by series expansions the motion of a string with given initial position and initial velocity under the influence of its stress only. This theory is still rather laborious and not especially well fit for the solution of impact problems.

The simplest impact problem, the longitudinal impact at the end of an infinitely long straight cable or bar, has been solved directly in 1868 by B. de Saint Venant (2) who determined the stress which is produced in the cable for the case of a constant impact velocity by a simple formula.

* See References 1, 3, 4 and 5

But it was not before 1948 that similar simple formulas for the perpendicular and oblique impact with slight approximations have been derived by the author of the present monography. These formulas have been used since in this country and abroad, especially for arresting gear studies and for further theoretical developments. The impact formulas can be used for the stepwise solution of any cable dynamics problem, if the number of required steps is not too large to be practical. This method of approach to problems of cable dynamics which moreover reflects in the best way the physics of the phenomena involved is closely related to the method of characteristics which has been successfully used also in other fields of physics and seems to be the simplest way of approach to the problems in question. It also has been followed by other authors in this field and will be applied in this monography too.

SYMBOLS

E	Modulus of elasticity of cable (lbs/ft^2)
ρ	Mass density of cable material ($lbs.sec^2/ft^4$)
ρ_o	Mass density of cable material at zero stress ($lbs.sec^2/ft^4$)
q	Cross section area of cable (ft^2)
l, l_o	Cable lengths (ft)
T	Tension (lbs)
σ	Stress (lbs/ft^2)
T_o	Initial tension (lbs)
σ_o	Initial stress (lbs/ft^2)
$\varepsilon, \varepsilon_o$	Elongation (ft)
x, y	Rectangular coordinates in plane of cable motion (ft)
s	Abscissa of a cable point in initial position along x-axis at time $t = 0$ (ft)
t	Time (sec)
θ	Angle between cable element and x-axis
c	Longitudinal wave velocity (ft/sec)
c_o	Longitudinal wave velocity at zero stress (ft/sec)
$\bar{c}$	Transverse wave velocity (ft/sec)
V_o	Impact velocity (ft/sec)
u	Particle velocity (ft/sec)
ω	Kink-velocity (ft/sec)
β	Impact angle
$H, H_o, \cdots$	Energies (ft/lbs)
$\eta, \eta_o, \cdots$	Dimensionless energy coefficients

CHAPTER I: BASIC CONCEPTIONS AND RELATIONS

1. The Elastic String

By an elastic string, we mean a circular cylinder of homogenous material whose diameter is small compared with its length, which is completely flexible, is elongated if a load is applied longitudinally, and which assumes its original length if the load is removed.

The load by which the string is elongated is called its tension T measured usually in pounds.

With the elongation of the string due to a load, a lateral contraction is normally combined. Therefore, the cross section area q decreases with increasing load. The true stress σ of the string is defined by the relation

$$\sigma = \frac{T}{q} \tag{1}$$

which is the load per unit cross section area. In technical considerations, the stress is usually defined by

$$\sigma = \frac{T}{q_o} \tag{2}$$

where q_o is the original cross section area belonging to the load zero. The true stress is, therefore larger than the stress plainly. For materials which do not elongate very much under high loads, the difference between q and q_o is generally negligibly small so that there is no remarkable difference between the two definitions of the stress. This is not the case, however, for very extensible materials.

In the following we will use the technical definition (2) of the stress, but before that we will discuss some relations which permit us to estimate the error resulting from the assumption of a constant cross section area $q = q_o$.

2. Hooke's Law of Elasticity

We denote by l_0 the length of a straight elastic string with the initial stress zero (see Figure 1). Then Hooke's law states that to an elongation ε_0 corresponds a stress σ_0 which is proportional to that elongation. If the factor of proportionality is denoted by $\frac{E}{l_0}$ we have

$$\frac{\sigma_0}{E} = \frac{\varepsilon_0}{l_0} \tag{3}$$

The constant E is called the modulus of elasticity.

Actually this statement is no general law, but a characterization of the material. It is very well satisfied by most metals.

Now if l is the length of the same string having the stress σ_0 then

$$l = l_0 + \varepsilon_0 \tag{4}$$

Elongating this string about another amount ε we obtain a stress which is determined by

$$\frac{\sigma}{E} = \frac{\varepsilon_0 + \varepsilon}{l_0} \tag{5}$$

Eliminating from the equations (3) + (5) the values l_0 and ε_0 we obtain

$$\boxed{\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \frac{\varepsilon}{l}} \tag{6}$$

which is the expression for Hooke's law if the string has initially the length l and the stress σ_0.

If $\frac{\sigma_0}{E}$ is negligibly small compared with 1 which is the case for metals and steel cables with hemp core formula (6) can be simplified to

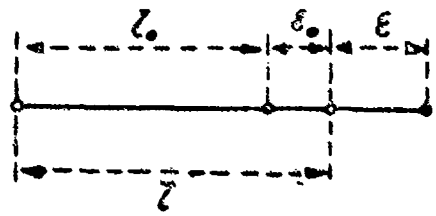


FIGURE 1

$$\frac{\sigma - \sigma_0}{E} = \frac{\varepsilon}{l} . \tag{7}$$

3. Lateral Contraction

We consider an elastic string with the initial stress $\sigma_0 = 0$ and assume that Hooke's law is valid. From the expression (6) for Hooke's law follows then:

$$\frac{\sigma}{E} = \frac{\varepsilon}{l} \tag{8}$$

where l is now the length of the string with zero stress, ε its elongation and σ the resulting stress. It is generally assumed that the lateral contraction is proportional to the elongation and, therefore, to the stress. If this is actually the case, and if we denote by d the diameter of the string at zero stress and with the negative value δ, its contraction due to the load we get

$$\frac{\delta}{d} = -\frac{1}{m}\frac{\sigma}{E} \tag{9}$$

where $-\frac{1}{m}$ is the factor of proportionality. $\frac{1}{m}$ is called Poisson's ratio.

σ is the true stress. We denote now with $\bar{\sigma}$ the stress related to the initial cross section area q_0 at zero stress. Then

$$q_0 = \left(\frac{d}{2}\right)^2\pi \quad , \qquad q = \left(\frac{d+\delta}{2}\right)^2\pi$$

and

$$\frac{q}{q_0} = \left(1 + \frac{\delta}{d}\right)^2. \tag{10}$$

Now

$$\sigma = \frac{T}{q} \quad , \qquad \bar{\sigma} = \frac{T}{q_0}$$

and because of (9) and (10)

$$\boxed{\frac{\bar{\sigma}}{E} = \frac{\sigma}{E}\left(1 - \frac{1}{m}\frac{\sigma}{E}\right)^2} \tag{11}$$

This shows that the simple stress $\bar{\sigma}$ is always smaller than the true stress σ. For metals and steel wire cables with hemp core the value $\frac{1}{m}\frac{\sigma}{E}$ is negligibly small compared with 1 for all stresses smaller than the braking stresses. In these cases, therefore, $\bar{\sigma} = \sigma$ is always a very good approximation. For other materials and cables, the error made by setting $\bar{\sigma} = \sigma$ can be estimated from this formula.

In general, the volume of the string varies with the load. In the present case, the initial volume under assumption of zero stress is

$$V_0 = q_0 l$$

and the volume at the stress σ corresponds to the elongation

$$V = q(l + \varepsilon).$$

Therefore

$$\frac{V}{V_0} = \frac{q}{q_0}\left(1 + \frac{\varepsilon}{l}\right) \tag{12}$$

We denote by ρ the mass density of the string material at the stress σ and especially with ρ_0 the mass density of the string at zero stress. Then

$$\frac{V}{V_0} = \frac{\rho_0}{\rho}$$

and because of (12), (10) and (8)

$$\frac{\rho_0}{\rho} = \left(1 - \frac{1}{m}\frac{\sigma}{E}\right)^2\left(1 + \frac{\sigma}{E}\right). \tag{13}$$

If $\frac{\sigma}{E}$ is small compared with 1, this stress density relation yields

$$\frac{\rho_0}{\rho} \approx 1 + \left(1 - \frac{2}{m}\right)\frac{\sigma}{E}. \tag{14}$$

Therefore, the density is constant with varying σ if $\frac{1}{m} = 0.5$. For small extensions, rubber satisfies this relation with good approximation ($\frac{1}{m} \approx 0.48$). For metals, the value of $\frac{1}{m}$ is mostly near 0.3. The cross section area q is constant with varying σ if $\frac{1}{m} = 0$, a case which is never accurately realized in nature. In this case, the stress density relation (14) yields

$$\boxed{\frac{\rho_0}{\rho} = 1 + \frac{\sigma}{E}} \tag{15}$$

4. The Cable as Elastic String

In the following a cable will always be considered an elastic string. However, a cable has not a homogenous structure. It consists of strands of uniform or frequently of different materials. It has to be defined, therefore, in which way such cable has to be replaced by an equivalent elastic string. Since the elastic string will be in any case only an approximation with respect to the characteristic properties of the cable, such definition will, to a certain degree, always be an arbitrary one.

For the following investigations we have in mind cables consisting of strands of a homogeneous material, mostly metal wire and metal wire cables with a core of hemp or a similar supporting material. Of main interest for the dynamics of such cable are its elastic and its inertia properties, while its geometrical structure is less important. The elastic properties are determined mainly by the metal strands. We relate, therefore, stresses to the metallic cross section area and call this q. Since, however, we do not wish to neglect the mass of the hemp or other supporting material if such is present, we define the mass density of the equivalent string by the relation

$$\boxed{\rho = \frac{W}{q \cdot g}} \tag{16}$$

where W is the weight in pounds of the cable per foot length and $g = 32.2$ ft/sec^2 the acceleration due to gravity while q is the metallic cross section area (ft^2). As mentioned before q will be considered as constant with varying loads so that the following investigations concern the ideal case of a Poisson's ratio of $\frac{1}{m} = 0$.

5. Cable Data

The following, Table 1, contains some data for a steel wire cable with hemp core which contains 6 strands with 19 wires each. Measurements which are discussed in this monography have been done mostly with cables of this series.

TABLE 1

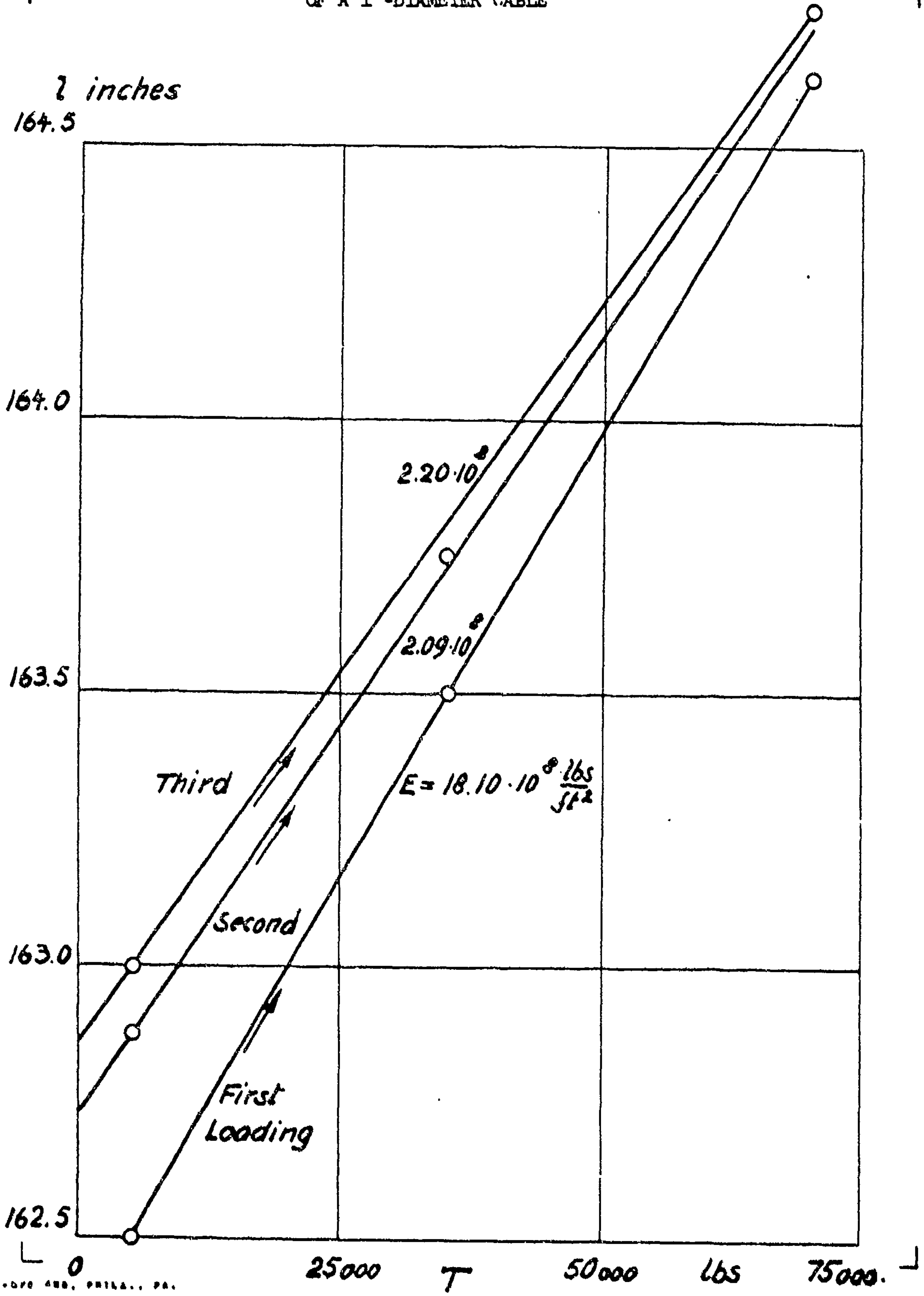
Nominal Diameter Inches	Maximum Pitch Inches	Metallic Area Ft^2	Weight per foot lbs/ft	Breaking Strength T_{max} lbs	Mass Density $lbs/sec^2/ft^4$	Elasticity Modulus lbs/ft^2
11/16	4.5	0.00119	0.71	46000	18.51	
7/8	5.7	0.00210	1.23	73000	18.20	
1	6.5	0.00274	1.60	95000	18.12	
1-1/8	7.3	0.00346	2.03	119000	18.20	Average
1-1/4	8.1	0.00428	2.50	146000	18.13	Value
1-3/8	8.9	0.00519	3.03	175000	18.12	$18.3 \cdot 10^8$
1-1/2	9.8	0.00617	3.60	208000	18.10	
1-5/8	10.6	0.00725	4.[illegible]3	242000	18.10	
1-3/4	11.4	0.00838	4.90	280000	18.16	
1-7/8	12.2	0.00962	5.63	314000	18.19	
2	13.0	0.01094	[illegible].40	350000	18.16	

The elasticity moduli E in this table have been measured at relaxed cables.* The ratio $\frac{\sigma_{max}}{E}$ where σ_{max} is the breaking stress has for all these cables a value near 0.0183.

Figure 2 shows the lengths of a 1" cable repeatedly loaded and measured for different loads. It proves that Hooke's law of elasticity is very well satisfied up to loadings near the breaking strength.

* Compare References 18, 19.

FIGURE 2
MEASUREMENT OF THE ELASTICITY MODULUS
OF A 1"-DIAMETER CABLE



Chapter II. Longitudinal Motion of a Cable

1. Mathematical Description of the Longitudinal Cable Motion

We consider a cable which is initially situated along a straight line and has a constant initial stress σ_0. We choose this straight line as x-axis of a one dimensional coordinate system with a point O as origin. The motion of the cable is restricted to a motion in this line. The initial time is $t = 0$. At this time any cable point P has a certain x-coordinate which we denote by s (see Figure 3). This point moves with the cable motion and has at a time $t > 0$ a position P'.

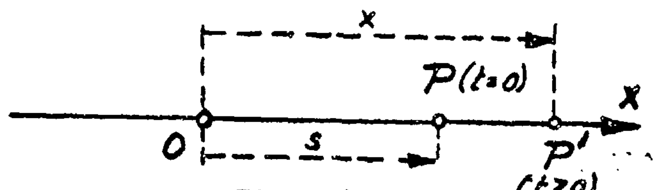


Figure 3

The coordinate of this point is denoted by x. Any other cable point P with a different coordinate s at the time $t = 0$ moves in the same time into another position with a different x-coordinate. In general, the motion of the total cable is described by the x-values which belong to the initial s values at any time t, with other words by a function

$$x = f(s, t). \tag{17}$$

At the time $t = 0$ the point P' coincides with P. Therefore, the function $f(s, t)$ has to satisfy the condition

$$f(s, 0) = 0 \tag{18}$$

for all values of s. For instance, the function $x = s + ct$ describes a longitudinal translation of the cable each point s moving with the constant velocity c toward the right if c is positive and to the left if c is negative.

Any arbitrary function $x = f(s, t)$ which satisfies the condition (18) represents actually a longitudinal cable motion which can be produced physically by moving each cable point so as the function prescribes it. This, however, requires adequate outside forces acting on all points of the cable. If such outside forces are applied only at certain distinct points of the cable, the motion of the other cable points cannot anymore be prescribed arbitrarily but will follow automatically due to the tension in the cable produced by moving certain points arbitrarily. This shows that if a cable or cable part moves under the influence of tension forces only the function $x = f(s, t)$ will have to satisfy additional conditions which have to be derived in the following.

2. Equations of Motion and Stress

We consider an element of the cable moving longitudinally under the influence of the tension within the cable only (see Figure 4). At the time $t = 0$ the element is situated between two points with the coordinates s and $s+\Delta s$. At the time t = const. >0 it is situated between the points x and $x + \Delta x = x + \frac{\partial x}{\partial s}\Delta s$.

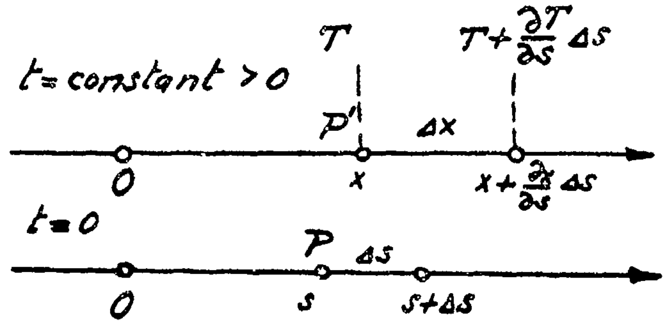


Figure 4

The mass of the element is in both positions the same and equal $q \cdot \Delta s \cdot \rho$ where ρ is the mass density of the cable material under the initial stress σ_o = const. and q the cross section area which is supposed to be constant during the motion.

At the time t = const. a tension difference moves the element, the tension at x being equal T and at $x + \Delta x$ equal $T + \Delta T = T + \frac{\partial T}{\partial s}\Delta s$. The moving force is, therefore, equal $\frac{\partial T}{\partial s}\Delta s$. The elongation at the time t = const. is then determined by Newton's law which yields

$$q \cdot \Delta s \cdot \rho \frac{\partial^2 x}{\partial t^2} = \frac{\partial T}{\partial s}\Delta s$$

or

$$\rho \frac{\partial^2 x}{\partial t^2} = \frac{\partial \sigma}{\partial s}. \tag{19}$$

This equation connects the two unknown functions x and σ of the variables s and t.

Another condition for these two functions is obtained from Hooke's law. The original length of the cable element $l = \Delta s$. Its elongation due to the stress σ is $\varepsilon = \Delta x - \Delta s = \left(\frac{\partial x}{\partial s} - 1\right)\Delta s$. Hooke's law in the form of equation (6) yields, therefore

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \frac{\partial x}{\partial s} - 1 \tag{20}$$

which is another equation connecting the two unknown functions x and σ of s and t.

If we eliminate σ from both equations (19) and (20) differentiating at first equation (20) with respect to s and replacing afterwards $\frac{\partial \sigma}{\partial s}$ by the left side of equation (19) we obtain

$$\boxed{\frac{\partial^2 x}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial s^2}} \tag{21}$$

where

$$\boxed{c^2 = \left(1 + \frac{\sigma_0}{E}\right)\frac{E}{\rho}} \tag{22}$$

<u>This is the condition which the function x = f (s, t) has to satisfy if it describes the longitudinal motion of a cable under the influence of tension forces only.</u>

If x = f (s, t) is a special solution of equation (21) the stress belonging to this solution follows from equation (20).

If $\frac{\sigma_0}{E}$ is small compared with 1 between the density ρ at the stress σ_0 and the density ρ_0 at zero stress the relation

$$\frac{\rho_0}{\rho} = 1 + \frac{\sigma_0}{E}$$

had been derived (equation (15)). Using ρ_0 instead of ρ in formula (22) we obtain

$$\boxed{c^2 = \left(1 + \frac{\sigma_0}{E}\right)^2 \frac{E}{\rho_0}} \tag{23}$$

3. Solution of the Wave Equation and the Longitudinal Wave Velocity

Equation (21) is known under the name of wave equation which is suggested by the physical meaning of its solution. Its general solution is given by

$$x = F(s+ct) + G(s-ct) \tag{24}$$

where F and G are arbitrary functions of the arguments s + ct respectively s - ct which possess two derivatives with respect to these arguments. That x is a solution of equation (21) can be easily verified by differentiation. We have

$$\frac{\partial x}{\partial s} = F' + G' \quad , \qquad \frac{\partial x}{\partial t} = c(F' - G')$$

$$\frac{\partial^2 x}{\partial s^2} = F'' + G'' \quad , \qquad \frac{\partial^2 x}{\partial t^2} = c^2(F'' + G'')$$

and therefore

$$\frac{\partial^2 x}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial s^2}.$$

Here the primes denote the derivatives with respect to the total argument of the function, for instance

$$F' = \frac{dF}{d(s+ct)} \quad , \qquad G' = \frac{dG}{d(s-ct)}$$

In order to find the physical meaning of the solution (24) we consider the special case

$$x = F(s+ct)$$

with a positive value of the constant c. For t = 0 this solution yields x = F(s) (We do not satisfy yet the former condition x = s for t = 0). In Figure 5 the function x = F (s) is plotted in a rectangular (s, x) - coordinate system. We consider a point P of this curve belonging to the abscissa s. In Figure 5 is further plotted the curve $x_1 = F(s_1 + ct)$ for a constant value of the time t. We consider the point P_1 of this curve which belongs to the value $s_1 = s - ct$ of the variable s_1.

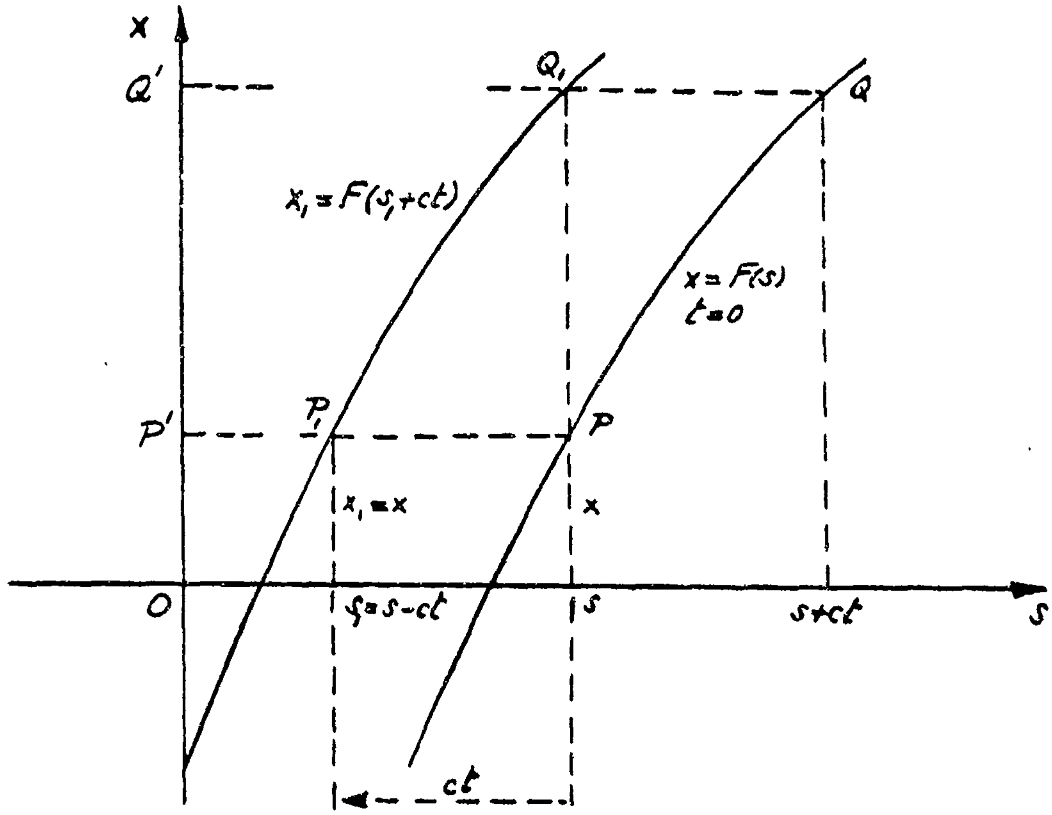


FIGURE 5

Then

$$x_1 = F(s_1 + ct) = F(s - ct + ct) = F(s) = x$$

This shows that the curve $x_1 = F(s_1 + ct)$ is a translation of the curve in negative s-direction about the amount ct. If the time t increases the curve $x = F(s)$ moves parallel to itself in this direction with the velocity c. Such motion is called a wave motion and c the wave velocity. The point P given by $x = F(s)$ for a constant s at the time $t = 0$ moves during the time t into the position Q_1 determined by $x = F(s + ct)$. If we project P and Q on the x-axis which represents the cable we get the points P' and Q'. Here is Q' the position of the cable point P' after the time t due to the cable motion. It should be noted that with the wave motion the point P does not move parallel to the s-axis into the position P but upwards into the position Q_1.

In the same way the solution $x = G(s - ct)$ represents a wave motion projected on the x-axis with the opposite direction of motion but with the same speed c.

c is called the longitudinal wave velocity for the cable motion. But it must be remembered that it is a velocity related to the variable s. The value of c has been determined by equation (23).

The general solution (24) of the wave equation consists accordingly in two waves running in opposite directions with the same speed c. We can now satisfy the condition (18) that $x = s$ for $t = 0$. The two arbitrary functions in equation (24) are then reduced to one only by the condition

$$s = F(s) + G(s)$$

Thus

$$G(s-ct) = s - ct - F(s-ct)$$

and, therefore

$$x = F(s+ct) - F(s-ct) + s - ct \tag{25}$$

F is an arbitrary function of the argument ξ if we denote by ξ either $s + ct$ or $s - ct$. We can write also

$$F(\xi) = \phi(\xi) + \frac{\xi}{2}$$

where $\phi(\xi)$ is another arbitrary function of ξ.

Then equation (25) takes the form

$$x = \phi(s+ct) + \frac{s+ct}{2} - \left(\phi(s-ct) + \frac{s-ct}{2}\right) + s - ct$$

or

$$\boxed{x = \phi(s+ct) - \phi(s-ct) + s} \tag{26}$$

This function x of s and t where ϕ is any arbitrary function and $c = \sqrt{\frac{E}{\rho_0}}$ describes a longitudinal motion of an elastic cable and any such motion for which x = s for t = 0 can be described in this form.

We denote by u the velocity of a cable element and get

$$u = \frac{\partial x}{\partial t} = c\left(\phi'(s+ct) + \phi'(s-ct)\right)$$

where the prime denotes the derivate with respect to the total argument of the function. Thus we find:

The velocity u of a cable particle is determined by

$$\boxed{\frac{u}{c} = \phi'(s+ct) + \phi'(s-ct)} \qquad (27)$$

From equation (20) follows finally the result:

The stress σ at any point s at any time t is determined by

$$\boxed{\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \phi'(s+ct) - \phi'(s-ct)} \qquad (28)$$

4. The Longitudinal Impact

We apply the general result of the preceding section contained in the formulas (26), (27) and (28) to the case of the longitudinal impact. We assume for this purpose that a cable with infinite length is situated along the negative x-axis ending in 0 and having the initial stress σ_o = constant at the time $t = 0$. Immediately afterwards point 0 moves suddenly with the constant velocity v_o in positive x-direction (see Figure 6).

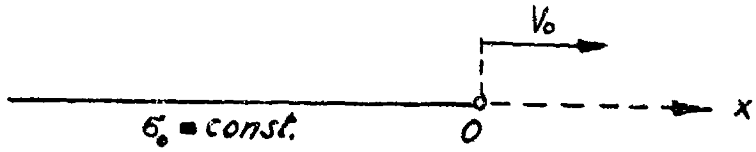


FIGURE 6

In the initial position the s-values of the cable points satisfy the condition $s \leqq 0$. Since at this time $t = 0$ the cable is in rest formula (27) yields

$$\phi'(s) = 0 \qquad \text{for } s \leqq 0 \tag{29}$$

For $t > 0$ point 0 ($s = 0$) moves with the velocity v_o. Thus formula (27) gives

$$\phi'(ct) + \phi'(-ct) = \frac{v_o}{c} \qquad \text{for } t > 0 \tag{30}$$

Because $-ct$ is negative $\phi'(-ct) = 0$ according to (29).

Therefore

$$\phi'(ct) = \frac{v_o}{c} \qquad \text{for } t > 0 \tag{31}$$

or for any variable ζ according to (29) and (31)

$$\phi'(\zeta) = 0 \text{ for } \zeta \leqq 0, \qquad \phi'(\zeta) = \frac{v_o}{c} \quad \text{for } \zeta > 0 \tag{32}$$

In the general relation (27)

$$\frac{u}{c} = \phi'(s+ct) + \phi'(s-ct)$$

the variable $s - ct$ is always $\leqq 0$, therefore, always

$$\phi'(s-ct) = 0$$

and

$$\frac{u}{c} = \varphi(s+ct) = 0 \quad \text{for} \quad s+ct \leqq 0$$
$$= \frac{v_0}{c} \quad \text{for} \quad s+ct > 0 \tag{33}$$

In the same way follows for the stress σ :

$$\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \phi'(s+ct) = 0 \quad \text{for} \quad s+ct \leqq 0 \tag{34}$$
$$= \frac{v_0}{c} \quad \text{for} \quad s+ct > 0$$

The stress σ induced by the impact is, therefore, constant and given by

$$\boxed{\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \frac{v_0}{c}} \tag{35}$$

The formulas (33) and (34) show that the stress given by (35) propagates with the longitudinal wave velocity c toward the left and the points under stress are moving with the velocity v_0 while beyond the point $s = -ct$ the cable is still in rest (see Figure 7).

FIGURE 7

If $\sigma_0 = 0$ formula (35) yields

$$\frac{\sigma}{E} = \frac{v_0}{c} \tag{36}$$

Because of (22) c is given then by $c^2 = \frac{E}{\rho}$ and equation (36) can be written also in the form

$$\sigma = v_0 \sqrt{E\rho} \tag{37}$$

which is B. de Saint Venant's formula derived in 1868 (*).

If $\frac{\sigma_0}{E}$ is negligibly small compared with 1, the longitudinal impact formula (35) can be simplified to

$$\boxed{\frac{\sigma-\sigma_0}{E} = \frac{v_0}{c}} \tag{38}$$

* Reference 2.

The general result (35) can be derived in an elementary way if the longitudinal wave velocity c is known. It is physically rather obvious that the impact with a constant velocity v_o (see Figure 7) must produce a constant stress especially if the cable is considered as a chain of small masses and massless springs. While now during the time t point 0 moves about $v_o t$ the stress σ propagates up to the point -ct. Beyond -ct the cable is in rest. Thus an original cable length l = ct has been elongated about the amount $\varepsilon = v_o t$. According to Hooke's law in the form (6) is, therefore,

$$\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}} = \frac{v_o}{c}$$

Of interest is the energy balance for the case of a longitudinal impact with a constant velocity v_o.

The work which is done at the cable end 0 moving it during the time t against the tension T = $q\sigma$

$$H = q\sigma \cdot v_o t \tag{39}$$

This energy is divided in two parts, the kinetic energy H_m of the moving cable segment and the strain energy H_σ in this segment. We have

$$H_m = q\rho c t \cdot \frac{v_o^2}{2} \tag{40}$$

since the length of the moving cable segment is ct and, therefore, $q\rho c t$ its mass and

$$H_\sigma = \frac{\sigma+\sigma_o}{2}\, q\, v_o t \tag{41}$$

because the strain energy is the work necessary to elongate the segment. The force required for this work increases linearly from $\sigma_o q$ to σq. Its mean value is $\frac{\sigma+\sigma_o}{2} q$. The path on which this work is done is the elongation $v_o t$. It must be now

$$H = H_m + H_\sigma$$

which yields

$$\sigma = \rho c \frac{v_o}{2} + \frac{\sigma + \sigma_o}{2}$$

or

$$\frac{\sigma - \sigma_o}{E} = \frac{\rho c^2}{E} \cdot \frac{v_o}{c} \tag{42}$$

The formula (22) for c^2 shows that

$$\frac{\rho c^2}{E} = 1 + \frac{\sigma_o}{E} \tag{43}$$

Therefore, the formula (42) is identical with (35) and proves that the energies are correctly balanced.

From the elementary derivation of the longitudinal impact formula (35) and the energy balance inversely follows the formula

$$c^2 = \left(1 + \frac{\sigma_o}{E}\right) \frac{E}{\rho} \tag{44}$$

for the longitudinal wave velocity c.

From (40) and (41) we obtain the relation

$$\frac{H_\sigma}{H_m} = \frac{\sigma + \sigma_o}{\rho c^2 \cdot \frac{v_o}{c}}$$

If we replace here ρc^2 by $E\left(1 + \frac{\sigma_o}{E}\right)$ from formula (44) and $\frac{v_o}{c}$ by $\frac{\sigma - \sigma_o}{E} \Big/ 1 + \frac{\sigma_o}{E}$ from the longitudinal impact formula (35) we find

$$\boxed{\frac{H_\sigma}{H_m} = \frac{\sigma + \sigma_o}{\sigma - \sigma_o}} \tag{45}$$

which shows that for zero initial stress

$$H_m = H_\sigma$$

so that the total energy is equally split in kinetic-and strain energy.

For $\sigma_o \neq 0$ the strain energy is always larger than the kinetic energy.

5. Longitudinal Motion and Stress of an Infinite Cable

A cable of infinite length is situated initially at the time $t = 0$ along the negative x-axis ending in point 0 and having the initial stress $\sigma_0 =$ const. Beginning with the time $t = 0$ point 0 is supposed to move with a given velocity in positive x-direction where

$$v_0 = f_0(t)$$

is a given function of the time t.

This problem is the same as that of the preceding section except that now v_0 is variable with the time. The determination of the unknown function $\phi(s, t)$ is done in complete analogy to the longitudinal impact problem. At the time $t = 0$ the velocity of each cable point $s \leqq 0$ is zero. Thus from (27)

$$\phi'(s) = 0 \text{ for } s \leqq 0 \tag{46}$$

For $t > 0$ point 0 ($s = 0$) moves with the velocity $v_0 = f_0(t)$. Thus from (27)

$$\phi'(ct) + \phi'(-ct) = \frac{f_0(t)}{c} \quad \text{for } t > 0$$

and since $-ct$ negative

$$\phi'(ct) = \frac{f_0(t)}{c} \quad \text{for } t > 0 \tag{47}$$

For any variable ξ, therefore,

$$\phi'(\xi) = 0 \quad \text{if} \quad \xi \leqq 0, \qquad \phi'(\xi) = \frac{f_0\left(\frac{\xi}{c}\right)}{c} \quad \text{if } \xi > 0 \tag{48}$$

The velocity u generally is consequently

$$\begin{aligned} \frac{u}{c} &= 0 && \text{for } \frac{s}{c} + t \leqq 0 \\ &= \frac{1}{c} f_0\left(\frac{s}{c} + t\right) && \frac{s}{c} + t > 0 \end{aligned} \tag{49}$$

according to formula (27) and the stress given by

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \frac{u}{c} \tag{50}$$

according to formula (28). If we define the function $f_0(\xi)$ for negative ξ to be zero then always

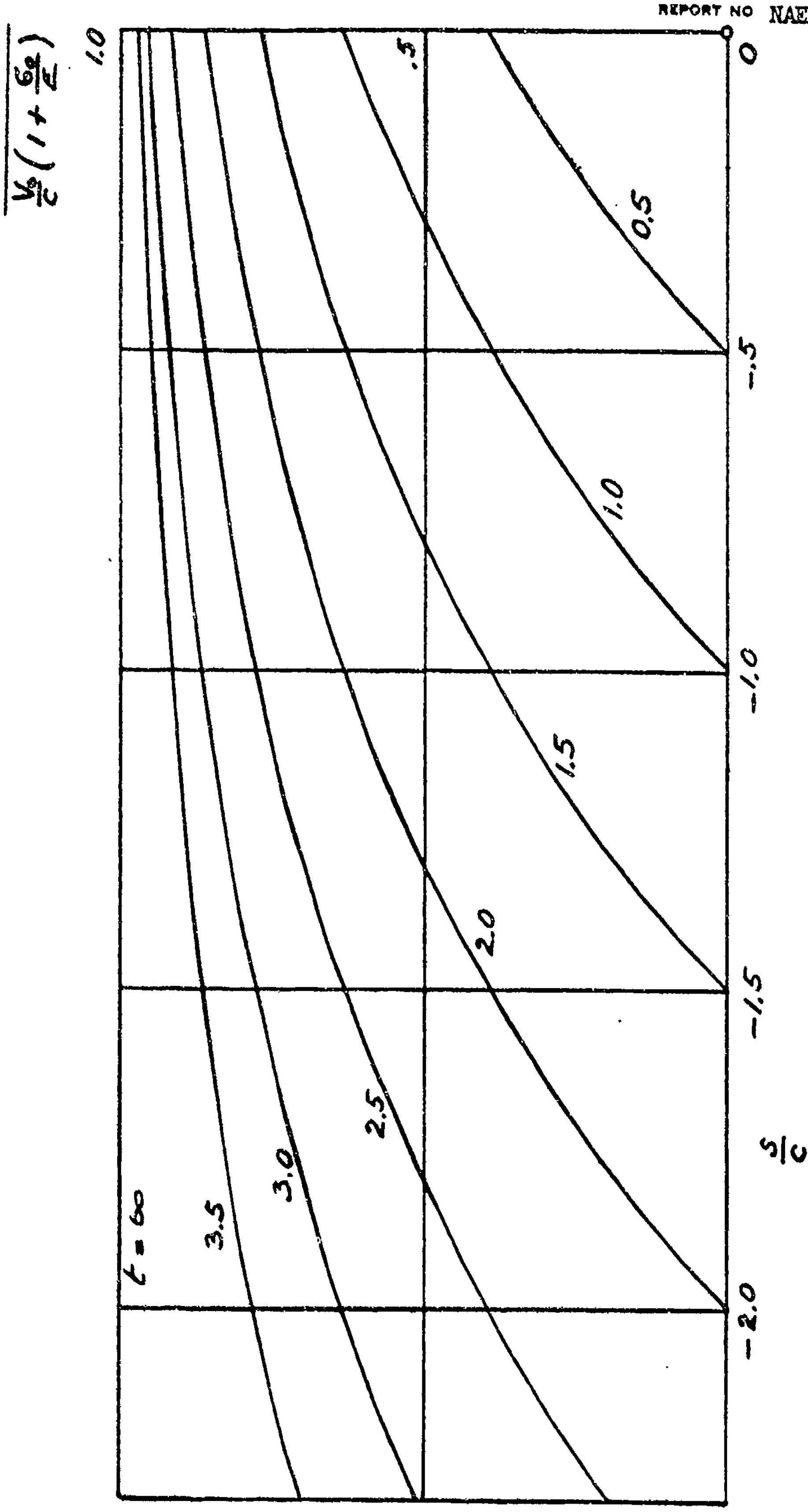

$\dfrac{\frac{\sigma-\sigma_0}{E}}{\frac{V_0}{c}\left(1+\frac{\sigma_0}{E}\right)}$
1.0
.5
0
-.5
-1.0
-1.5
-2.0
$\frac{s}{c}$
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$t = \infty$

Fig. 8

$$\boxed{\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \frac{1}{c} f_0\left(\frac{s}{c}+t\right)} \tag{51}$$

For a constant $v_0 = f_0$ this formula is identical with the longitudinal impact formula (35).

If $\frac{s}{c}$ is small compared with t formula (51) yields

$$\frac{\sigma-\sigma_0}{E} \Big/ 1+\frac{\sigma_0}{E} = \frac{1}{c}\left[f_0(t) + f_0'(t)\frac{s}{c} + \cdots\right]$$

or under restriction to the first power of $\frac{s}{c}$:

$$\boxed{\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \frac{v_{s=0}}{c} + \frac{\dot{v}_{s=0}}{c^2}\, s} \tag{52}$$

where $v_{s=0}$ is the velocity and $\dot{v}_{s=0}$ the acceleration of the cable end point.

The last result shows that the stress near the cable end point 0 is increasing toward the cable end point if the acceleration is positive and decreasing if the acceleration is negative.

Of interest is an example in which the velocity of the cable end point 0 increases from zero approaching asymptotically a constant value v_0 .

We choose accordingly $f_0(t) = v_0(1-e^{-t})$

The stress σ which develops in this case follows from formula (51) which yields

$$\frac{\sigma-\sigma_0}{E} \Big/ 1+\frac{\sigma_0}{E} = \frac{v_0}{c}\left(1-e^{-\left(\frac{s}{c}+t\right)}\right)$$

In Figure 8 ... are plotted the values of $\frac{\sigma-\sigma_0}{E} \Big/ \frac{v_0}{c}\left(1+\frac{\sigma_0}{E}\right)$ versus $\frac{s}{c}$ for different values of the time. $\frac{s}{c}$ represents the cable in its initial position shortened by the factor $\frac{1}{c}$. It is interesting to see that the stress increases with increasing velocity $f_0(t)$ and if the value v_0 is reached (for $t = \infty$) assumes the impact stress value belonging to v_0 . This indicates that the stress maximum is independent of the acceleration but depends only on the final velocity.

6. Longitudinal Motion and Stress of a Finite Cable

We assume that a finite cable is situated initially in rest between the values $s = 0$ and $s = l$ and has the initial stress $\sigma_0 =$ constant (see Figure 9). It is assumed that for $t > 0$ the end point $s = 0$ moves with the velocity $V_1 = f_1(t)$ and the end point $s = l$ with the velocity $V_2 = f_2(t)$.

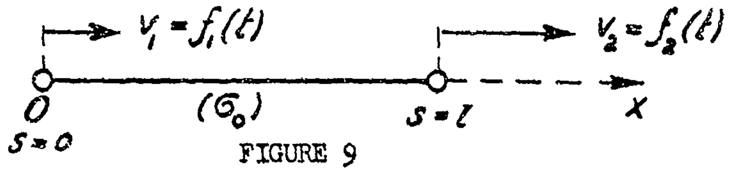


FIGURE 9

For the determination of the function $\phi = \phi(s,t)$ for this motion we use again the formula (27) $\frac{u}{c} = \phi'(s+ct) + \phi'(s-ct)$ as in the case of the cable with infinite length. At the time $t = 0$ the velocity of the cable point s where $0 \leqq s \leqq l$ is zero. Therefore

$$\phi'(s) = 0 \quad \text{for} \quad 0 \leqq s \leqq l.$$

For $t > 0$ the point $s = 0$ moves with the velocity $V_1 = f_1(t)$ and the point $s = l$ with the velocity $V_2 = f_2(t)$. Therefore,

$$\left.\begin{aligned} \phi'(ct) + \phi'(-ct) &= \frac{1}{c} f_1(t) \\ \phi'(l+ct) + \phi'(l-ct) &= \frac{1}{c} f_2(t) \end{aligned}\right\} t > 0$$

For any variable ζ we have accordingly

$$\phi'(\zeta) = 0, \qquad 0 \leqq \zeta \leqq 0 \tag{53}$$

$$\left.\begin{aligned} \phi'(\zeta) + \phi'(-\zeta) &= \frac{1}{c} f_1\left(\frac{\zeta}{c}\right) = h_1(\zeta) \\ \phi'(l+\zeta) + \phi'(l-\zeta) &= \frac{1}{c} f_2\left(\frac{\zeta}{c}\right) = h_2(\zeta) \end{aligned}\right\} \zeta > 0 \qquad \begin{aligned} (54) \\ (55) \end{aligned}$$

The last three equations determine the function $\phi'(\zeta)$ for all values of ζ. The first equation (53) determines ϕ' in the interval $0 \leqq \zeta \leqq 0$. For these values of ζ the term $\phi'(\zeta)$ in the second equation (54) is determined (equal zero). Thus

$$\phi'(-\xi) = h_1(\xi) \quad \text{for} \quad l \geqq \xi > 0$$

This equation determines $\phi'(\xi)$ within the interval

$$-l \leqq \xi < 0$$

In the third equation (53) the function $\phi'(l-\xi)=0$ for ξ-values $0 \leqq \xi \leq l$.

Thus $$\phi'(l+\xi) = h_2(\xi) \quad \text{for} \quad l \geqq \xi > 0$$

This equation determines $\phi'(\xi)$ within the interval

$$l < \xi \leqq 2l$$

Now in equation (53) $\phi'(\xi)$ is determined for $l < \xi \leqq 2l$.

We can write this equation

$$\phi'(l+\xi) + \phi'(-l-\xi) = h_1(l+\xi)$$

where now $\phi'(l+\xi)$ is determined for $l \geqq \xi > 0$. Thus

$$\phi'(-l-\xi) = h_1(l+\xi) - \phi'(l+\xi)$$

determines $\phi'(\xi)$ within the interval

$$-2l \leqq \xi < -l$$

Equation (54) can be written

$$\phi'(2l+\xi) + \phi'(-\xi) = h_2(l+\xi)$$

$\phi'(-\xi)$ is determined for $l \geqq \xi > 0$ Thus

$$\phi'(2l+\xi) = h_2(l+\xi) - \phi'(-\xi)$$

determines $\phi'(\xi)$ within the interval $2l < \xi \leqq 3l$

and so on.

The following table 2 contains the continuation of these recurrent formulas:

TABLE 2

The Equation	determines $\phi'(\xi)$ within the interval
$\phi'(\xi)=0$	$0 \leqq \xi \leqq l$
$\phi'(-\xi) = h_1(\xi)$	$-l \leqq \xi < 0$
$\phi'(l+\xi) = h_2(\xi)$	$l < \xi \leqq 2l$
$\phi'(-l-\xi) = h_1(l+\xi) - \phi'(l+\xi)$	$-2l \leqq \xi < -l$
$\phi'(2l+\xi) = h_2(l+\xi) - \phi'(-\xi)$	$2l < \xi \leqq 3l$
$\phi'(-2l-\xi) = h_1(2l+\xi) - \phi'(2l+\xi)$	$-3l \leqq \xi < -2l$
$\phi'(3l+\xi) = h_2(2l+\xi) - \phi'(-l-\xi)$	$3l < \xi \leqq 4l$
$\phi'(-3l-\xi) = h_1(3l+\xi) - \phi'(3l+\xi)$	$-4l \leqq \xi < -3l$
$\phi'(4l+\xi) = h_2(3l+\xi) - \phi'(-2l-\xi)$	$4l < \xi \leqq 5l$
$\phi'(-4l-\xi) = h_1(4l+\xi) - \phi'(4l+\xi)$	$-5l \leqq \xi < -4l$
$\phi'(5l+\xi) = h_2(4l+\xi) - \phi'(-3l-\xi)$	$5l < \xi \leqq 6l$
$\phi'(-5l-\xi) = h_1(5l+\xi) - \phi'(5l+\xi)$	$-6l \leqq \xi < -5l$
$\phi'(6l+\xi) = h_2(5l+\xi) - \phi'(-4l-\xi)$	$6l < \xi \leqq 7l$
$\phi'(-6l-\xi) = h_1(6l+\xi) - \phi'(6l+\xi)$	$-7l \leqq \xi < -6l$
$\phi'(7l+\xi) = h_2(6l+\xi) - \phi'(-5l-\xi)$	$7l < \xi \leqq 8l$
$\phi'(-7l-\xi) = h_1(7l+\xi) - \phi'(7l+\xi)$	$-8l \leqq \xi < -7l$
$\phi'(8l+\xi) = h_2(7l+\xi) - \phi'(-6l-\xi)$	$8l < \xi \leqq 9l$
$\phi'(-8l-\xi) = h_1(8l+\xi) - \phi'(8l+\xi)$	$-9l \leqq \xi < -8l$
- - - - - - - - - - - - - - -	- - - - - - - - - -

At the interval end points ϕ' has in general discontinuities.. If we eliminate successively the ϕ' values on the right sides of the equations of table 2 we find the result:

Within the intervals between the points $0, \pm l, \pm 2l, \cdots$ the function $\phi'(\xi)$ is determined by the equations

$$
\begin{aligned}
\phi'(\xi) &= 0 \\
\phi'(-\xi) &= h_1(\xi) \\
\phi'(l+\xi) &= h_2(\xi) \\
\phi'(-l-\xi) &= h_1(l+\xi) - h_2(\xi) \\
\phi'(2l+\xi) &= h_2(l+\xi) - h_1(\xi) \\
\phi'(-2l-\xi) &= h_1(2l+\xi) - h_2(l+\xi) + h_1(\xi) \\
\phi'(3l+\xi) &= h_2(2l+\xi) - h_1(l+\xi) + h_2(\xi) \\
\phi'(-3l-\xi) &= h_1(3l+\xi) - h_2(2l+\xi) + h_1(l+\xi) - h_2(\xi) \\
\phi'(4l+\xi) &= h_2(3l+\xi) - h_1(2l+\xi) + h_2(l+\xi) - h_1(\xi) \\
\phi'(-4l-\xi) &= h_1(4l+\xi) - h_2(3l+\xi) + h_1(2l+\xi) - h_2(l+\xi) + h_1(\xi) \\
\phi'(5l+\xi) &= h_2(4l+\xi) - h_1(3l+\xi) + h_2(2l+\xi) - h_1(l+\xi) + h_2(\xi) \\
\phi'(-5l-\xi) &= h_1(5l+\xi) - h_2(4l+\xi) + h_1(3l+\xi) - h_2(2l+\xi) + h_1(l+\xi) - h_2(\xi) \qquad (56) \\
\phi'(6l+\xi) &= h_2(5l+\xi) - h_1(4l+\xi) + h_2(3l+\xi) - h_1(2l+\xi) + h_2(l+\xi) - h_1(\xi) \\
\phi'(-6l-\xi) &= h_1(6l+\xi) - h_2(5l+\xi) + h_1(4l+\xi) - h_2(3l+\xi) + h_1(2l+\xi) \\
&\quad - h_2(l+\xi) + h_1(\xi) \\
\phi'(7l+\xi) &= h_2(6l+\xi) - h_1(5l+\xi) + h_2(4l+\xi) - h_1(3l+\xi) + h_2(2l+\xi) \\
&\quad - h_1(l+\xi) + h_2(\xi)
\end{aligned}
$$

- -

and so on where

$$0 < \xi < l,$$

$$h_1(\xi) = \frac{1}{c} f_1\left(\frac{\xi}{c}\right), \qquad h_2(\xi) = \frac{1}{c} f_2\left(\frac{\xi}{c}\right). \tag{57}$$

The velocity u of any cable point s at any time t is then determined by

$$\frac{u}{c} = \phi'(s+ct) + \phi'(s-ct) \tag{58}$$

and the stress σ by

$$\frac{\sigma-\sigma_0}{E} \Big/ 1+\frac{\sigma_0}{E} = \phi'(s+ct) - \phi'(s-ct) \tag{59}$$

This result shows that the problem in question requires addition or subtraction of values of the given functions (57) only for its solution. These functions, determining the velocities of the cable end points can be given, therefore, also graphically, no analytical expression being necessary for the determination of the velocity and the stress as functions of s and t. The determination of x (equation 26) requires an integration which can be performed graphically too.

Example 1:

We consider the case of

$$v_1 = f_1(t) = 0, \qquad v_2 = f_2(t) = v_0 = \text{constant}$$

which represents the longitudinal impact at a finite cable where one end point is fixed and the other end point suddenly moves with the constant velocity v_0. In this case

$$h_1(\xi) = 0, \qquad h_2(\xi) = \frac{v_0}{c}$$

are both constants. The equations (56) yield

$$\begin{array}{lll}
\phi'(\xi) = 0 & \phi'(3l+\xi) = 2\frac{v_0}{c} & \phi'(5l+\xi) = 3\frac{v_0}{c} \\
\phi'(-\xi) = 0 & \phi'(-3l-\xi) = -2\frac{v_0}{c} & \phi'(-5l-\xi) = -3\frac{v_0}{c} \\
\phi'(l+\xi) = \frac{v_0}{c} & \phi'(4l+\xi) = 2\frac{v_0}{c} & \phi'(6l+\xi) = 3\frac{v_0}{c} \\
\phi'(-l-\xi) = -\frac{v_0}{c} & \phi'(-4l-\xi) = -2\frac{v_0}{c} & \phi'(-6l-\xi) = -3\frac{v_0}{c} \\
\phi'(2l+\xi) = \frac{v_0}{c} & & ------ \\
\phi'(-2l-\xi) = -\frac{v_0}{c} & &
\end{array}$$

$$0 < \xi < l$$

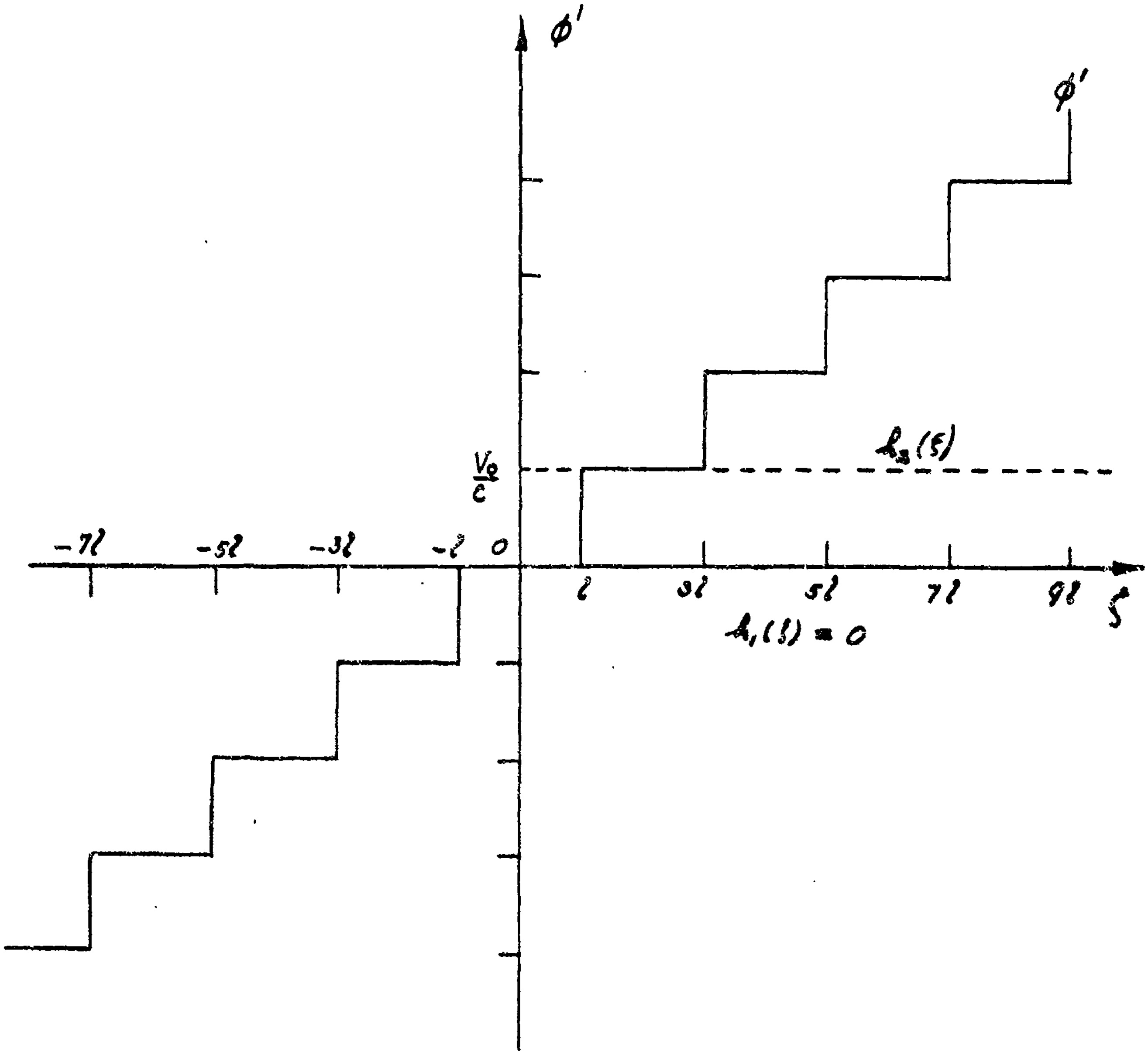


FIGURE 10

The function $\phi'(\zeta)$ for all values of ζ is, therefore, of the shape represented in Figure 10. Let us determine now, for instance, the velocity u and the stress σ in the cable point situated originally at $s = \frac{l}{2}$ at the time $t = \frac{l}{c}$. Then $ct = l$ and

$$s + ct = \frac{3}{2} l , \quad s - ct = -\frac{l}{2}$$

Thus because of (58) and (59) using the graph Figure 9

$$\phi'(s+ct) = \frac{v_0}{c} , \quad \phi'(s-ct) = 0$$

$$u = v_0 ,$$

$$\frac{\sigma - \sigma_0}{E} = \left(1 + \frac{\sigma_0}{E}\right) \frac{v_0}{c} .$$

Figure 10 shows the stress development over the length $\frac{s}{l}$ as function of $\frac{ct}{l}$. At the cable end $\frac{s}{l} = 1$ the longitudinal impact stress σ_1, determined by

$$\frac{\frac{\sigma_1 - \sigma_0}{E}}{1 + \frac{\sigma_1}{E}} = \frac{v_0}{c} ,$$

builds up at the time $t = 0$. The stress difference $\sigma_1 - \sigma_0$ between σ_1 and the initial stress σ_0 propagates

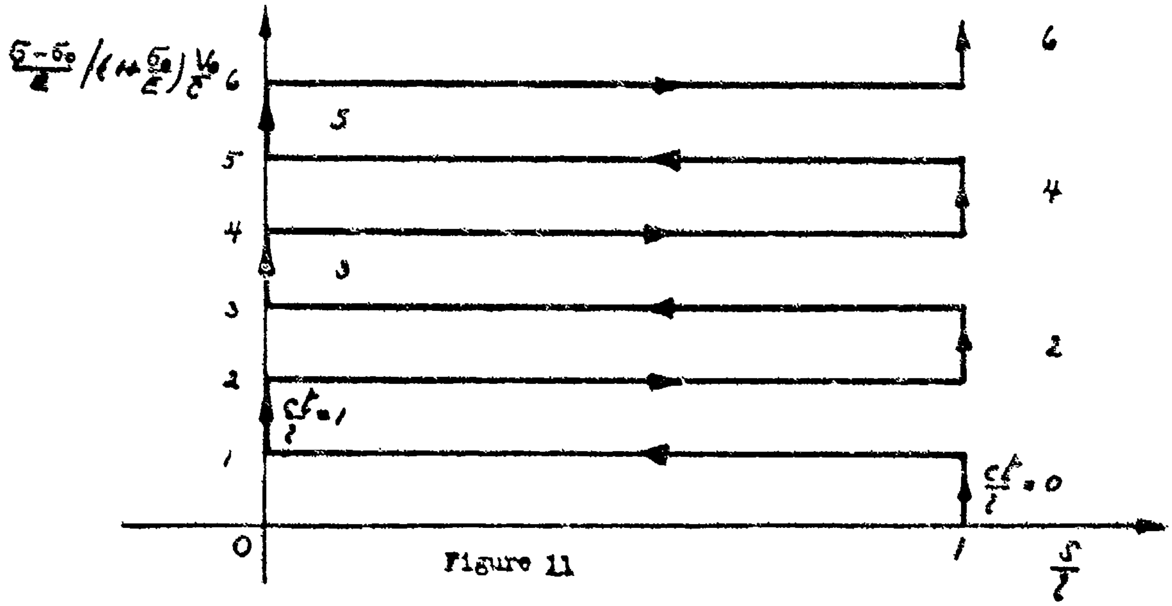


Figure 11

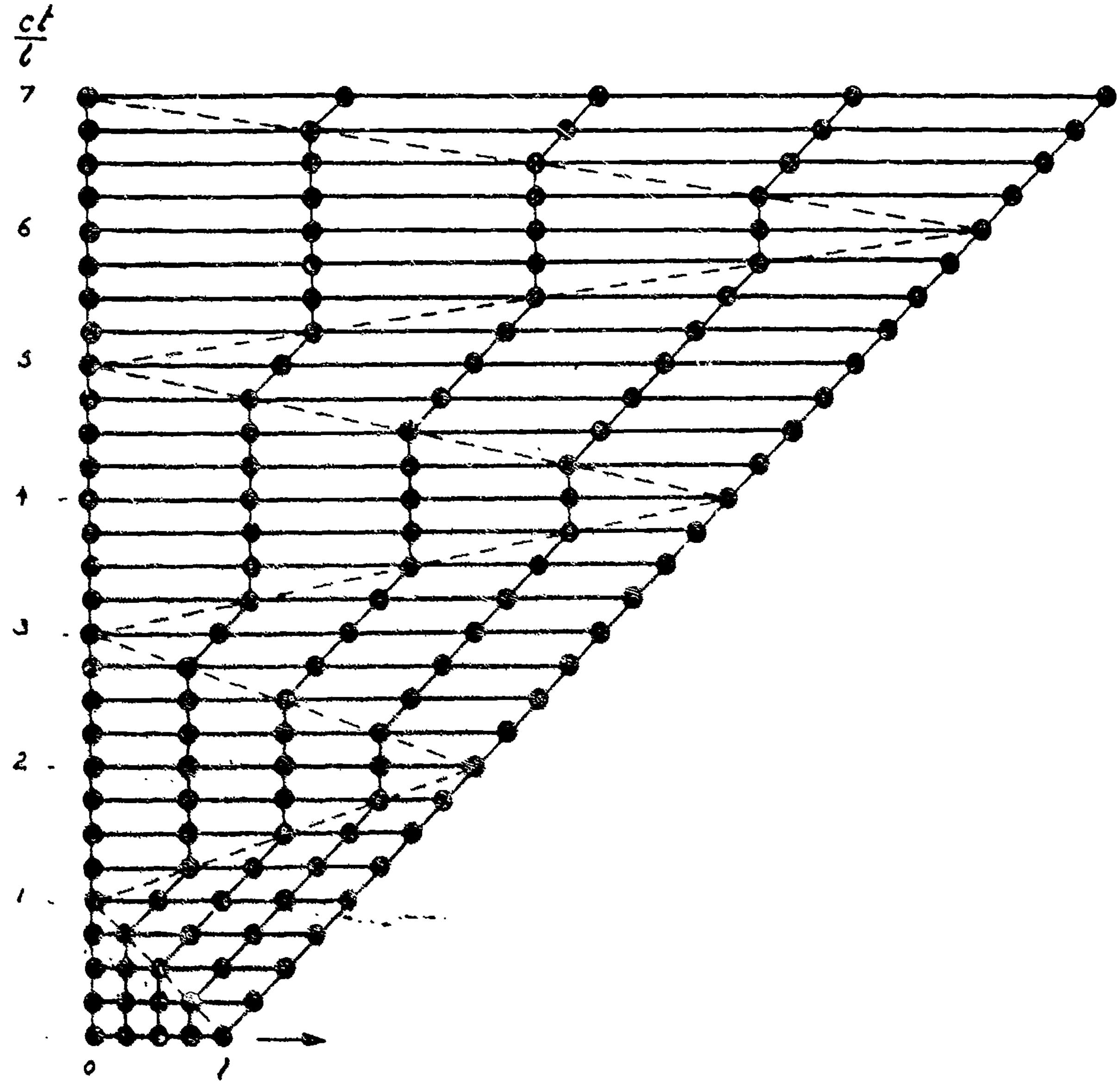


FIGURE 12

along the cable toward O and is there reflected completely. Up to this moment the stress value $\sigma = \sigma_1$ is constant. After the reflection we have the stress $\sigma = \sigma_1 + (\sigma_1 - \sigma_0) = 2\sigma_1 - \sigma_0$. Indeed from Figure 9 follows, for instance, at $s = 0$ and ct $= \frac{3}{2} l$

$$\phi'(s+ct) = \frac{v_0}{c}, \qquad \phi'(s-ct) = -\frac{v_0}{c}$$

Thus

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = 2\frac{v_0}{c}$$

If we substract

$$\frac{\frac{\sigma_1 - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \frac{v_0}{c}$$

we obtain

$$\frac{\frac{\sigma - \sigma_1}{E}}{1 + \frac{\sigma_0}{E}} = \frac{v_0}{c} = \frac{\frac{\sigma_1 - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}}$$

or

$$\sigma - \sigma_1 = \sigma_1 - \sigma_0.$$

The stress difference $\sigma_1 - \sigma_0$ is reflected once more at the end point $\frac{s}{l} = 1$ at the time t for which $\frac{ct}{l} = 2$ and so on as demonstrated by Figure 11. In Figure 12 the motion of the cable points is explained for this case. Five mass points of the cable are considered at different times for which $\frac{ct}{l} = 0, .25, .50, \ldots\ldots$. For the illustration the cable is moving upward with the speed $\frac{c}{l}$ while the right end point is moving with a constant speed to the right. The figure shows how the elongation starts at the right cable and propagates toward the left until the left cable end is reached. Then a new elongation starts at the left and and propagates toward the right and so on.

This example contains the general proof that a <u>propagating stress is completely reflected at a fixed cable end point.</u>

The reflection at a moving cable end point follows from equation (51). In this case the reflected stress for a fixed end point is superposed by the stress produced by the longitudinal impact with the velocity of the moving cable end point.

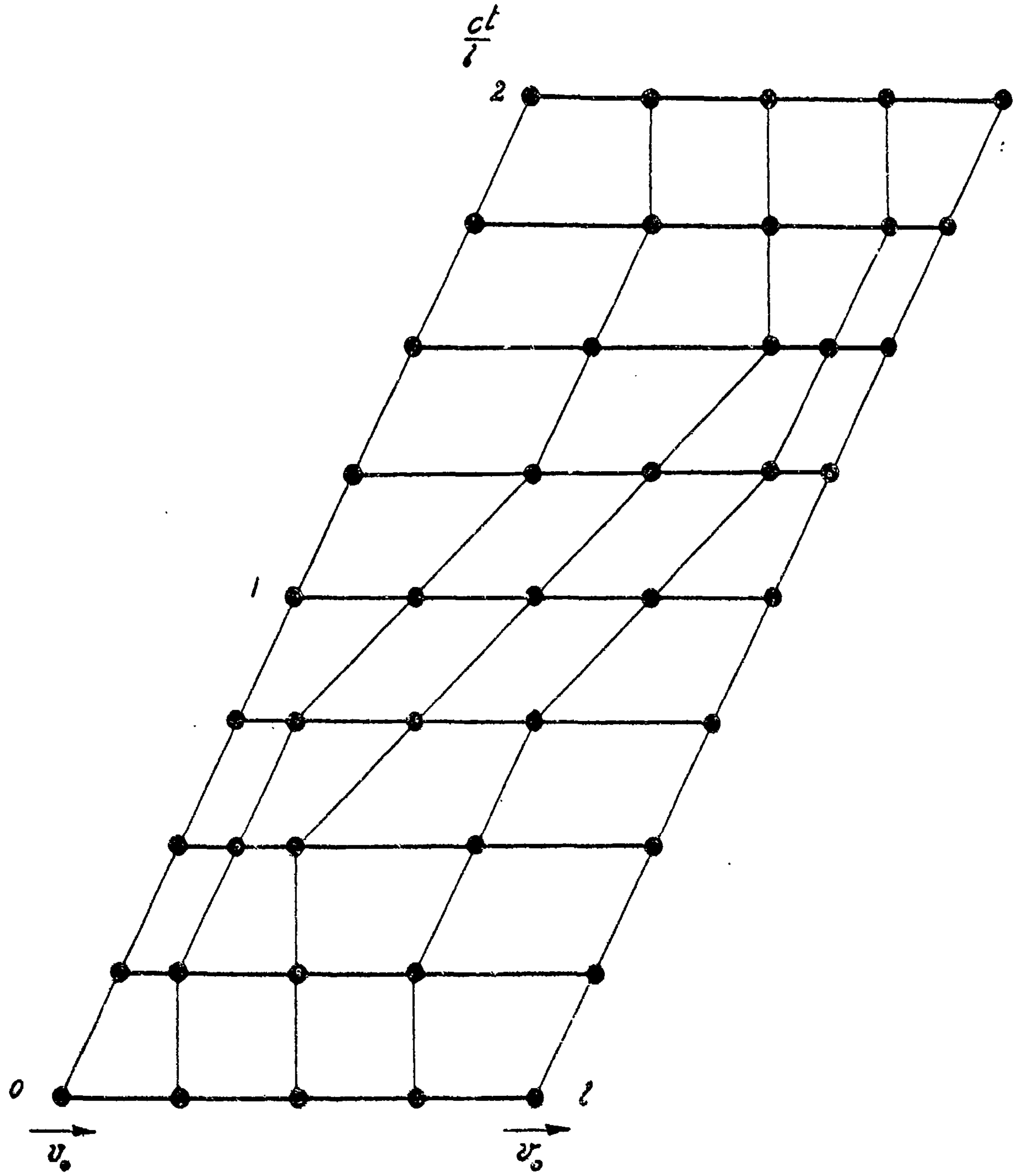


FIGURE 1b

This impact velocity is negative if the motion tends to shorten the cable, otherwise it is positive. Accordingly the superposing stress is negative or positive.

Example 2:

We consider the case

$$v_1 = v_2 = v_0 = \text{constant}$$

Then $\phi'(\xi) = 0, \quad \phi'(-\xi) = \frac{v_0}{c}, \quad \phi'(l+\xi) = \frac{v_0}{c}, \quad \phi'(-l-\xi) = 0,$

$\phi'(2l+\xi) = 0, \quad \phi'(-2l-\xi) = \frac{v_0}{c}, \quad \phi'(3l+\xi) = \frac{v_0}{c}, \quad \phi'(-3l-\xi) = 0,$

Figure 13 shows the function $\phi'(\xi)$. Figure 14 demonstrates the motion of five mass points of the cable. The point $\sigma = \frac{l}{2}$ has always the initial stress

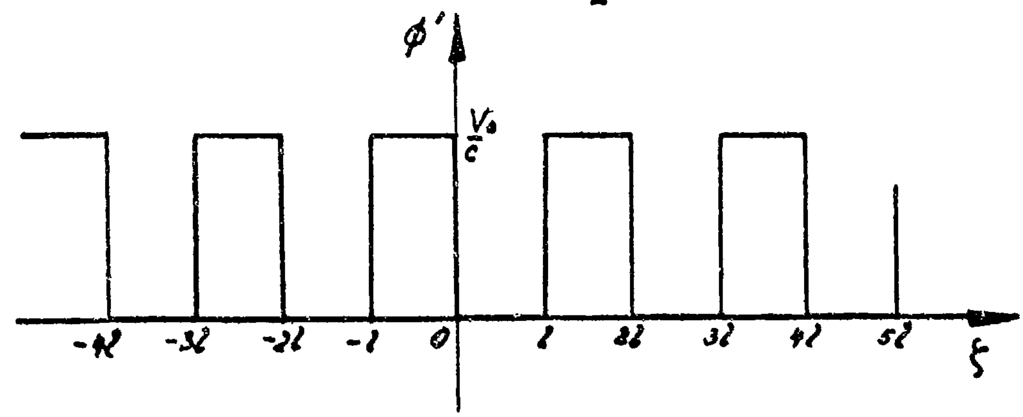


Figure 13

Example 3: Uniform longitudinal motion of a cable.

Under a uniform motion will be understood a motion for which $\phi'(\xi) = 0$ for $\xi \leqq l$ (see Figure 15).

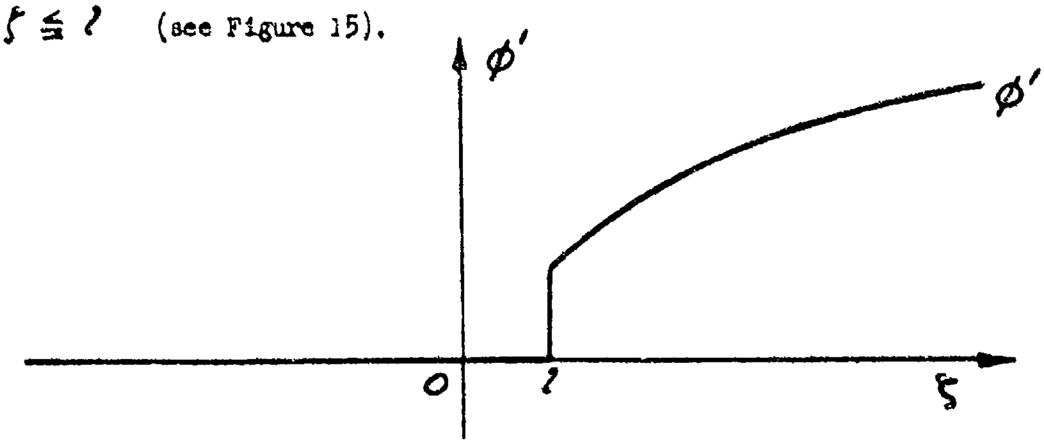


Figure 15

The formulas for the velocity and the stress (58) and (59) yield in this case

$$\frac{\sigma-\sigma_0}{E} \Big/ 1+\frac{\sigma_0}{E} = \frac{u}{c} = \phi'(s+ct)$$

The cable is situated initially between the points s = 0 and s = l and therefore, for each cable point s the value $\phi'(s-ct) = 0$. Each cable point - one after the other -passes through the same velocity or stress distribution with respect to time. The velocities of the cable end points are $v_1 = f_1(t) = c\,\phi'(ct)$, $v_2 = f_2(t) = c\,\phi'(l+ct)$.

If one of these functions is prescribed the function ϕ' is completely determined and so is the other of the two functions. If the prescribed function for instance $f_2(t)$ is free of vibrations the resulting function $f_1(t)$ determines the motion of the end point s = 0 required for a vibration-free motion of the total cable.

7. Approximate Formulas for the Motion and Stress of a Finite Cable

From the result of the preceding section, we derive now some simple approximate formulas for the velocity and the stress of a longitudinally moving finite cable. We write formula (58) in a slightly different form

$$\frac{u}{c} = \phi'\left(\frac{s}{c}+t\right) + \phi'\left(\frac{s}{c}-t\right) \tag{60}$$

which can be done because ϕ' in formula (58) is an arbitrary function. If we write, for instance, $\phi'(s+ct) = \phi'\left(c\left(\frac{s}{c}+t\right)\right)$ then this is an arbitrary function of $\frac{s}{c}+t$ which we now denote by ϕ'.

Any function $\phi'(\xi)$ can be composed additionally by an odd and an even function of ξ so that

$$\phi'(\xi) = \phi_1(\xi) + \phi_2(\xi) \tag{61}$$

where the functions ϕ_1 and ϕ_2 satisfy the conditions

$$\phi_1(-\xi) = -\phi_1(\xi), \quad \phi_2(-\xi) = \phi_2(\xi). \tag{62}$$

For the derivatives of such functions hold the relations

$$\begin{aligned} \phi_1'(-\xi) &= \phi_1'(\xi), & \phi_2'(-\xi) &= -\phi_2'(\xi) \\ \phi_1''(-\xi) &= -\phi_1''(\xi), & \phi_2''(-\xi) &= \phi_2''(\xi) \end{aligned} \tag{63}$$

- -

Accordingly $\frac{u}{c}$ can be written in the form

$$\frac{u}{c} = \phi_1\left(\frac{s}{c}+t\right) + \phi_2\left(\frac{s}{c}+t\right) + \phi_1\left(\frac{s}{c}-t\right) + \phi_2\left(\frac{s}{c}-t\right). \tag{64}$$

We assume now that the finite cable is situated initially between the points $s = -\frac{l}{2}$ and $s = \frac{l}{2}$ and has the initial stress σ_0 = constant. After a certain time $\frac{s}{c}$ will be small compared with t and the functions in formula (64) can be expanded in Taylor series of powers of $\frac{s}{c}$. Restricting the expansion to terms of the first power we obtain approximately

$$\begin{aligned} \frac{u}{c} = {} & \phi_1(t) + \frac{1}{1!}\phi_1'(t)\frac{s}{c} + \phi_1(-t) + \frac{1}{1!}\phi_1'(-t)\frac{s}{c} \\ & + \phi_2(t) + \frac{1}{1!}\phi_2'(t)\frac{s}{c} + \phi_2(-t) + \frac{1}{1!}\phi_2'(-t)\frac{s}{c}. \end{aligned}$$

This yields because of the relations (62) and (63)

$$\frac{u}{c} = 2\left(\phi_2(t) + \phi_1'(t)\frac{s}{c}\right) \tag{65}$$

If now the velocities of the cable end points (see Figure 16) $s = -\frac{l}{2}$ and $s = \frac{l}{2}$ are prescribed for $t > 0$ by the functions

$$v_1 = f_1(t) \quad , \quad v_2 = f_2(t)$$

the two unknown functions in (65) must satisfy the condition

$$\frac{1}{2c} f_1(t) = \phi_2(t) - \phi_1'(t)\frac{l}{2c}$$

$$\frac{1}{2c} f_2(t) = \phi_2(t) + \phi_1'(t)\frac{l}{2c}$$

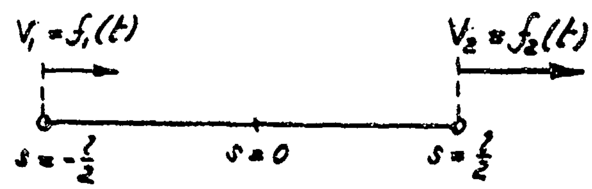


Figure 16

from which by subtraction and addition follows

$$\phi_1'(t) = \frac{1}{2l}\left(f_2(t) - f_1(t)\right) \tag{66}$$

$$\phi_2'(t) = \frac{1}{4c}\left(f_2(t) + f_1(t)\right) \tag{67}$$

If we substitute these values in formula (65) we obtain

$$\boxed{\frac{u}{c} = \frac{1}{2c}\left(f_2(t) + f_1(t)\right) + \frac{1}{l}\left(f_2(t) - f_1(t)\right)\frac{s}{c}} \tag{68}$$

which describes the velocity u of any cable point s $\left(-\frac{l}{2} \leq s \leq \frac{l}{2}\right)$ at any time approximately.

For the stress σ follows in the same way from the general formula

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \phi'\left(\frac{s}{c} + t\right) - \phi'\left(\frac{s}{c} - t\right)$$

the approximation

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = 2\left(\phi_1(t) + \phi_2'(t)\frac{s}{c}\right) \tag{69}$$

The values of $\phi_1(t)$ and $\phi_2'(t)$ can be computed by integration respectively differentiation from the formulas (66) and (67) so that

$$\boxed{\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \frac{1}{l}\int_0^t \left(f_2(t)-f_1(t)\right)dt + \frac{1}{2}\left(f_2'(t)+f_1'(t)\right)\frac{s}{c^2}} \tag{70}$$

The constant of integration has been chosen equal zero in this formula so that for $s=0$ and $t=0$ the stress $\sigma=\sigma_0$. It should be noted that the initial conditions $u=0$ and $\sigma=\sigma_0$ for $t=0$ cannot be satisfied anymore in general for all values s due to the approximation.

Formula (70) shows that at any time t the stress distribution over the cable is a linear function of s. The first term on the right side of formula (70) represents the term belonging to the static stress since the integral represents the elongation of the cable with the initial length l at the time t. The second term is a correction containing the mean acceleration $\frac{1}{2}\left(f_2'(t)+f_1'(t)\right)$ of the cable.

8. Variable Initial Conditions

In the preceding problems the cable was initially in rest $(u_0 = 0)$ having a constant initial stress σ_0. We consider now the case that initially the cable is already moving so that each cable point has an individual velocity u_0 which can be different at different cable points and that the initial stress σ_0 is variable from point to point too.

We assume that the cable started its motion out of rest and zero stress and denote by s the abscissa of a cable point in this condition. The actual initial position of this cable point where it has the velocity u_0 and the stress σ_0 we denote by x_0. Then u_0, σ_0, x_0 are functions of s

$$x_0 = x_0(s)\,, \quad u_0 = u_0(s)\,, \quad \sigma_0 = \sigma_0(s) \tag{71}$$

which represent the given initial conditions.

We denote by ρ the density of the cable material at zero stress.

The considerations used in section 2 yield now

$$\rho \frac{\partial^2 x}{\partial t^2} = \frac{\partial \sigma}{\partial s}$$

and

$$\frac{\sigma}{E} = \frac{\partial x}{\partial s} - 1$$

from which follows:

$$\frac{\partial^2 x}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial s^2}$$

where

$$c^2 = \frac{E}{\rho}$$

For $t = 0$ is now $x = x_0(s)$ instead of $x = s$ before. The general solution of the last differential equation is again

$$x = F(s+ct) + G(s-ct)$$

and therefore

$$\frac{u}{c} = F'(s+ct) - G'(s-ct)$$

$$\frac{\sigma}{E} = F'(s+ct) + G'(s-ct) - 1 .$$

For $t = 0$ we have because of the conditions (71)

$$x_0(s) = F(s) + G(s) \tag{72}$$

$$\frac{u_0(s)}{c} = F'(s) - G'(s) \tag{73}$$

$$\frac{\sigma_0(s)}{E} = F'(s) + G'(s) - 1 \tag{74}$$

We integrate equation (73) with respect to s and obtain together with (72)

$$F(s) = \frac{1}{2}\left(x_0(s) + \frac{1}{c}\int_0^s u_0(\xi)\,d\xi + K\right) \tag{75}$$

$$G(s) = \frac{1}{2}\left(x_0(s) - \frac{1}{c}\int_0^s u_0(\xi)\,d\xi - K\right) \tag{76}$$

where K is the constant of integration. The position of the cable point s at the time t is, therefore,

$$\boxed{x = \frac{1}{2}\left[x_0(s+ct) + x_0(s-ct) + \frac{1}{c}\int_{s-ct}^{s+ct} u_0(\xi)\,d\xi\right]} \tag{77}$$

a formula which has been derived at first by d'Alembert.

For the velocity u we obtain

$$\boxed{\frac{u}{c} = \frac{1}{2}\left[\frac{\sigma_0(s+ct) - \sigma_0(s-ct)}{E} + \frac{1}{c}\left(u_0(s+ct) + u_0(s-ct)\right)\right]} \tag{78}$$

For the stress follows:

$$\frac{\sigma}{E} = \frac{\partial x}{\partial s} - 1 = \frac{1}{2}\left(x_0'(s+ct) + \frac{1}{c}\,u_0(s+ct)\right) + \frac{1}{2}\left(x_0'(s-ct) - \frac{1}{c}\,u_0(s-ct)\right) - 1$$

and since

$$\frac{\sigma_0}{E} = x_0'(s) - 1$$

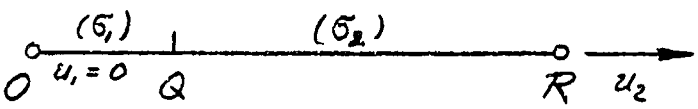
finally

$$\boxed{\frac{\sigma}{E} = \frac{1}{2}\left[\frac{\sigma_0(s+ct) + \sigma_0(s-ct)}{E} + \frac{u_0(s+ct) - u_0(s-ct)}{c}\right]} \tag{79}$$

The formulas (77), (78) and (79) determine the motion and the stress of the cable if the initial conditions (71) are given. The formulas show that these conditions are satisfied, indeed, if we set $t = 0$.

9. Reflection of a Longitudinal Wave at a Fixed End

We assume that the cable is in rest between the fixed endpoint 0 and point Q having here the stress σ_1. Between points Q and R the cable stress is assumed to be equal σ_2 produced by a longitudinal impact with the velocity u_2 at the cable in rest with the initial stress σ_1. Q is propagating toward 0. Now



$$\frac{\dfrac{\sigma_2 - \sigma_1}{E}}{1 + \dfrac{\sigma_1}{E}} = \frac{u_2}{c_1}$$

where

$$c_1 = \sqrt{\left(1 + \frac{\sigma_1}{E}\right)\frac{E}{\rho_1}}.$$

When Q has reached point 0 which is a fixed point a new impact occurs. If 0 would not be a fixed point it would move with the velocity u_2 to the right in the moment where Q coincides with 0. Since, however, 0 is a fixed cable point it acts on the cable segment QR with the relative velocity u_2 towards the left producing a stress σ_3 determined by

$$\frac{\dfrac{\sigma_3 - \sigma_2}{E}}{1 + \dfrac{\sigma_2}{E}} = \frac{u_2}{c_2}$$

where

$$c_2 = \sqrt{\left(1 + \frac{\sigma_2}{E}\right)\frac{E}{\rho_2}}$$

Now

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\sigma_1}{E}}{1 + \dfrac{\sigma_2}{E}}$$

Therefore

$$\frac{\dfrac{\sigma_2 - \sigma_1}{E}}{1 + \dfrac{\sigma_1}{E}} \sqrt{\left(1 + \frac{\sigma_1}{E}\right)\frac{E}{\rho_1}} = \frac{\dfrac{\sigma_3 - \sigma_2}{E}}{1 + \dfrac{\sigma_2}{E}} \sqrt{\left(1 + \frac{\sigma_2}{E}\right)\frac{E}{\rho_2}}$$

or

$$\frac{\sigma_3 - \sigma_2}{\sigma_2 - \sigma_1} = \sqrt{\frac{1 + \frac{\sigma_2}{E}}{1 + \frac{\sigma_1}{E}}} \sqrt{\frac{\rho_2}{\rho_1}}$$

or

$$\sigma_3 - \sigma_2 = \sigma_2 - \sigma_1 .$$

The incoming stress difference is completely reflected at the fixed endpoint.*

*Compare F. E. Marble, P. 47, where a different result is obtained.(ref 12)

CHAPTER III: Longitudinal Interaction Between Masses and Cables

1. Motion of a Mass by a Cable

We solve at first approximately the following problem. A mass attached to the end point O of a cable with the length ℓ, the cross section area q, the elasticity modulus E, the mass density ρ at zero initial stress is pulled by moving the other cable end point P with the constant velocity V_0. The motion of the mass and the stress in the cable have to be determined. The approximation in the solution will result from the simplifying assumption that the stress in the cable is constant along the cable length (but variable with the time).

We assume that the cable is situated along the x-axis between $x = 0$ and $x = \ell$ (see Figure 17) the mass m being attached to the cable at the end point $x = 0$. During the time t point P moves about $V_0 t$ while the mass m in O will describe a path x. Under the assumption of constant stress along the cable the stress follows from

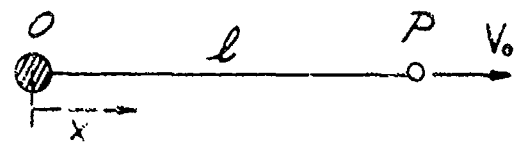


Figure 17

$$\frac{\sigma}{E} = \frac{V_0 t - x}{\ell} \tag{80}$$

$v_0 t - x$ being the elongation at the time t. The mass m is moved by the tension $T = q\sigma$. Therefore, according to Newton's law

$$m \frac{d^2 x}{dt^2} = m\ddot{x} = T = \frac{qE}{\ell}(v_0 t - x)$$

or

$$\ddot{x} + \frac{qE}{\ell m} x = \frac{qE V_0}{\ell m} t. \tag{81}$$

The mass m is assumed to be in rest at the time $t = 0$ so that at this time

$$x = 0, \quad \dot{x} = 0.$$

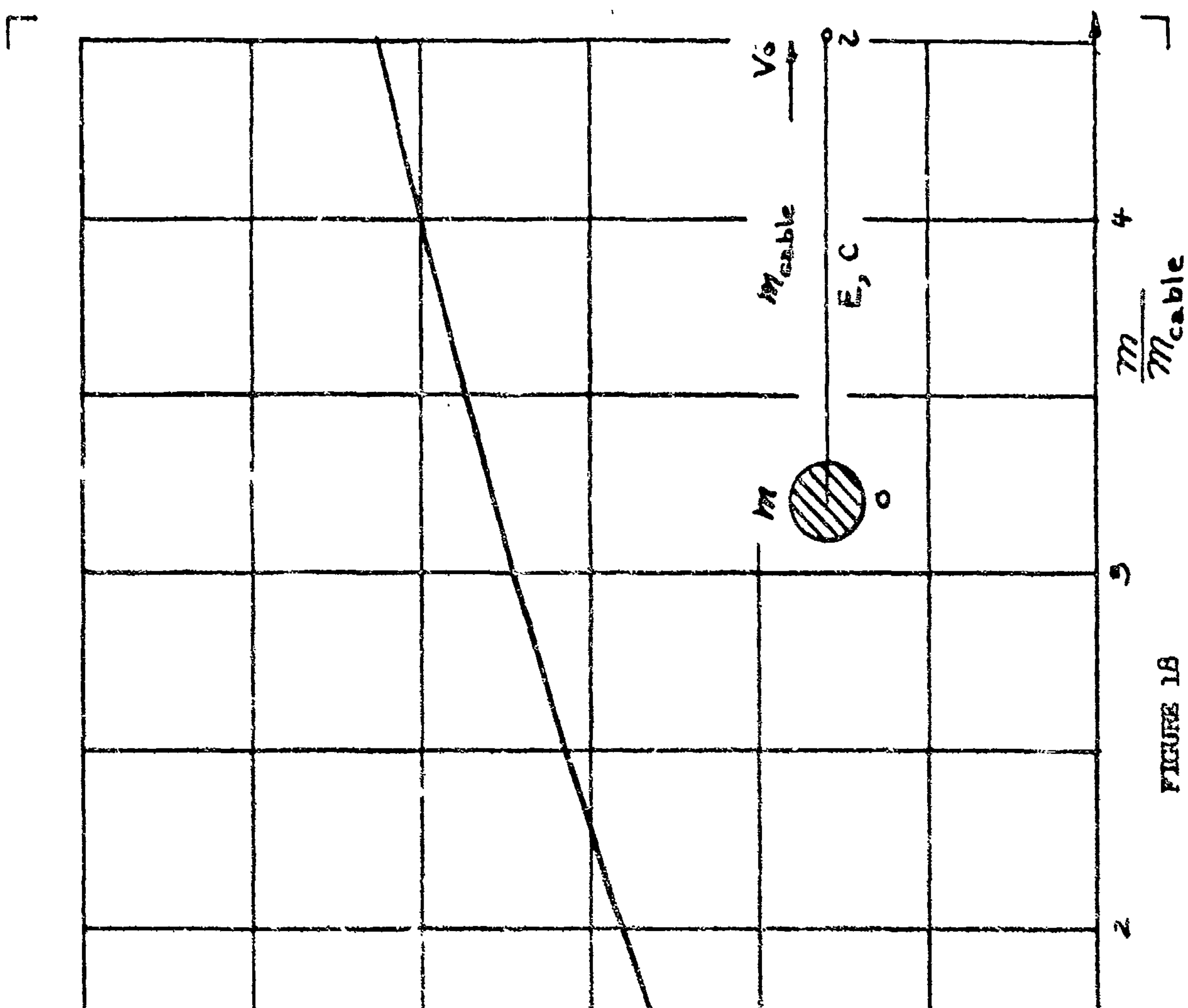


FIGURE 18
STRESS IN A CABLE WHICH PULLS A MASS

The solution of the differential equation (81) under these initial conditions is

$$x = v_0\left(t - \frac{1}{\omega}\sin\omega t\right) \tag{82}$$

where

$$\omega^2 = \frac{Eq}{\ell m} \tag{83}$$

This constant can be written also

$$\omega^2 = \frac{c^2}{\ell^2}\,\frac{m_{cable}}{m} \tag{84}$$

where $c = \sqrt{\frac{E}{\rho}}$ and $m_{cable} = \rho q \ell$ the constant c being the longitudinal wave velocity and m_{cable} the total mass of the cable. Substituting the value of x from (82) in formula (80) we obtain

$$\frac{\sigma}{E} = \frac{v_0}{c}\sqrt{\frac{m}{m_{cable}}}\,\sin\omega t. \tag{85}$$

This result shows that the stress vibrates as sin ωt, that its amplitude is proportional to $\frac{v_0}{c}$ and to the root of the mass ratio m/m_{cable}.

The stress maximum is reached at the time

$$t = \frac{\pi}{2\omega}$$

and is determined by

$$\frac{\sigma_{max}}{E} = \frac{v_0}{c}\sqrt{\frac{m}{m_{cable}}}. \tag{86}$$

This result is represented graphically in Figure 18. The velocity following from (82) is plotted in Figure 19.

The <u>exact solution</u> of the same problem can be derived from the general theory of Section II 6. The motion of the mass m is described by an unknown velocity function

$$v_1 = f_1(t)$$

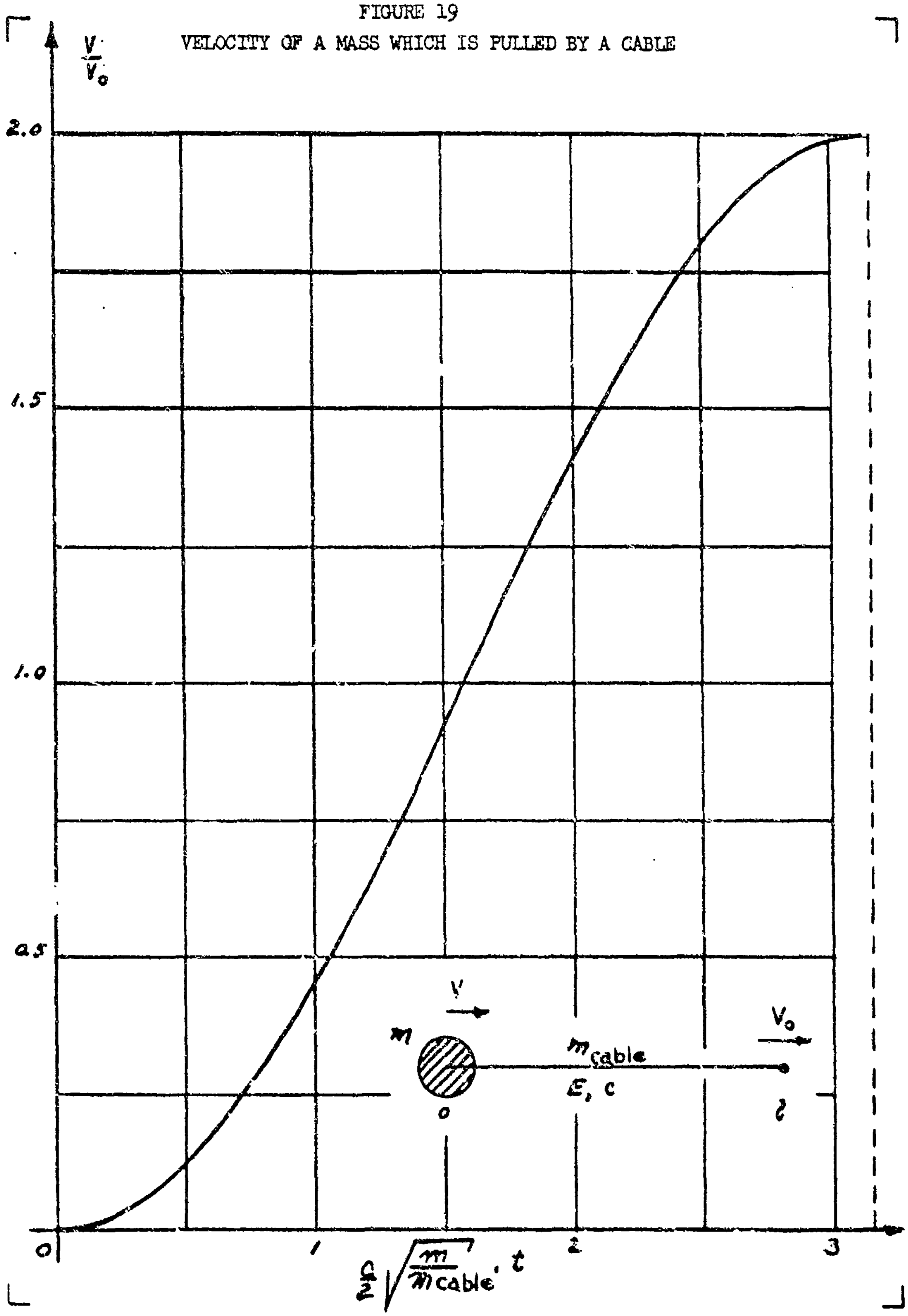
while the motion of the end point $s = \ell$ is given by

$$v_2 = f_2(t) = v_0 = constant.$$

Applying the general theory we have according to (57) (87)

$$h_2(\xi) = \frac{v_0}{c}.$$

FIGURE 19
VELOCITY OF A MASS WHICH IS PULLED BY A CABLE



The motion of the mass m is determined by Newton's law

$$m\dot{v}_1 = q\,\sigma_{\xi=0}$$

where

$$\frac{\sigma_{\xi=0}}{E} = \phi'(ct) - \phi'(-ct)$$

according to formula (59). Thus

$$m\dot{v}_1 = qE\left(\phi'(ct) - \phi'(-ct)\right). \tag{88}$$

The unknown function $\phi'(\xi)$ c... is connected with

$$h_1(\xi) = \frac{1}{c} f_1\left(\frac{\xi}{c}\right)$$

and $h_2(\xi) = \frac{V_0}{c}$ by the formulas (56). Setting

$$\xi = ct$$

we have

$$h_1(\xi) = \frac{1}{c} f_1(t)$$

and

$$\dot{v}_1 = \frac{d f_1(t)}{dt} = c^2 \frac{dh_1}{d\xi}.$$

Thus equation (88) can be written in the form

$$\frac{dh_1(\xi)}{d\xi} = \frac{qE}{mc^2}\left(\phi'(\xi) - \phi'(-\xi)\right). \tag{89}$$

We consider now the time interval for which

$$0 < \xi = ct < l.$$

For this interval the equations (56) show that

$$\phi'(\xi) = 0, \qquad \phi'(-\xi) = h_1(\xi).$$

Thus (89) yields

$$\frac{dh_1}{d\xi} = -\frac{qE}{mc^2} h_1$$

or

$$h_1 = C\,e^{-\frac{qE}{mc^2}\xi}$$

where C is a constant of integration. Because of $\xi = ct$ this equation is identical with

$$\frac{v_1}{c} = C\,e^{-\frac{qE}{mc}t}.$$

For $t \to 0$ we obtain

$$\frac{(V_1)_{t=0}}{c} = C.$$

The motion of m is assumed to start from rest. Therefore $C = 0$

and

$$h_1(\xi) = 0$$

or

$$\frac{V_1}{c} = 0 \quad \text{for} \quad 0 < \xi < l.$$

This shows that the mass m does not move until $ct = l$ in accordance with the fact that the disturbance due to the motion of the cable end point $s = l$ needs the time $t = \frac{l}{c}$ in order to reach the mass m.

We consider next the time interval for which

$$l < ct < 2l$$

If we replace ξ by $l + \xi$ the variable ξ is restricted again to

$$0 < \xi < l.$$

Equation (89) takes the form

$$\frac{d h_1(l+\xi)}{d(l+\xi)} = \frac{qE}{mc^2}\left(\phi'(l+\xi) - \phi'(-l-\xi)\right). \tag{90}$$

From the third and fourth equations (56) follows

$$\begin{aligned}\phi'(l+\xi) - \phi'(-l-\xi) &= 2h_2(\xi) - h_1(l+\xi)\\ &= 2\frac{V_0}{c} - h_1(l+\xi)\end{aligned}$$

and equation (90) yields

$$\frac{d h_1(l+\xi)}{d(l+\xi)} = \frac{qE}{mc^2}\left(2\frac{V_0}{c} - h_1(l+\xi)\right) \tag{91}$$

which is a linear differential equation for the unknown function h_1 of the variable $l + \xi$. Its general solution is

$$\frac{V_1}{c} = h_1(l+\xi) = 2\frac{V_0}{c} + C e^{-\frac{qE}{mc^2}(l+\xi)} \tag{92}$$

where C is the constant of integration. For $\xi = 0$ or $ct = l$ the mass m begins to move from rest. Therefore, $\frac{V_1}{c} = 0$ for $\xi = 0$

Equation (92) for $\xi = 0$ yields

$$\frac{V_1}{c} = 2\frac{V_0}{c} + C e^{-\frac{qE}{mc^2} l} = 0$$

Thus

$$C = -2\frac{V_0}{c} e^{\frac{qE}{mc^2} l}$$

and the solution (92) becomes

$$\frac{V_1}{c} = h_1(l+\xi) = 2\frac{V_0}{c}\left(1 - e^{-\frac{qE}{mc^2}\xi}\right) \tag{93}$$

where

$$ct = l + \xi,$$
$$0 < \xi < l$$

Next we consider the time interval for which

$$2l < ct < 3l$$

and replace ξ by $2l+\xi$ where now again $0 < \xi < l$.

Equation (89) takes the form

$$\frac{d\,h_1(2l+\xi)}{d(2l+\xi)} = \frac{qE}{mc^2}\left(\phi'(2l+\xi) - \phi'(-2l-\xi)\right).$$

From the fifth and sixth equation (56) follows

$$\phi'(2l+\xi) - \phi'(-2l-\xi) = 2h_2(l+\xi) - 2h_1(\xi) - h_1(2l+\xi).$$

We found $h_1(\xi) = 0$ for $0 < \xi < l$. Therefore

$$\frac{d\,h_1(2l+\xi)}{d(2l+\xi)} = \frac{qE}{mc^2}\left(2\frac{V_0}{c} - h_1(2l+\xi)\right). \tag{94}$$

This linear differential equation for h_1 as function of $2l+\xi$ has the same form as equation (91) and has the general solution

$$\frac{V_1}{c} = h_1(2l+\xi) = 2\frac{V_0}{c} + C e^{-\frac{qE}{mc^2}(2l+\xi)}. \tag{95}$$

The constant of integration C follows now from the consideration that the velocity V_1 for $\xi = 0$ must be the same as the velocity V_1 following from equation (93) for $\xi = l$. Thus

$$2\frac{V_0}{c} + C e^{-\frac{qE}{mc^2}\cdot 2l} = 2\frac{V_0}{c}\left(1 - e^{-\frac{qE}{mc^2} l}\right)$$

or

$$C = -2\frac{V_0}{c} e^{\frac{qE}{mc^2} l}$$

as before. The velocity of the mass m in the time interval

$$ct = 2l + \xi,$$
$$0 < \xi < l$$

is therefore determined by

$$\frac{V_1}{c} = h_1(2l+\xi) = 2\frac{V_0}{c}\left(1 - e^{-\frac{qE}{mc^2}(l+\xi)}\right). \tag{96}$$

Proceding further in the same way we obtain stepwise the exact solution of our problem. After $V_1 = f_1(t)$ has been determined the motion of any cable point and the stress in any point at any time follows from the consideration of section II 6.

The same procedure can be applied if the motion of the cable end point $S = l$ is given by an arbitrarily prescribed function

$$V_2 = f_2(t)$$

instead of being constant equal V_0.

2. Prescribed Acceleration of a Mass by Means of a Cable

The problem to be solved in this section is the opposite one to that of the preceding section. A mass m attached to a cable has to be accelerated in a prescribed manner by pulling it with a cable. The motion of the cable end point which produces the prescribed motion of the other cable end point where the mass is attached has to be determined.

We assume again that the cable is situated initially between the points $s = 0$ and $s = \ell$, that the initial stress is zero and that the mass m is attached to the end point $s = 0$ (see Figure 20). We denote the prescribed velocity of the mass in $s = 0$ by $u = u(t)$ and the unknown velocity of the end point $s = \ell$ by $v = v(t)$. According to the basic formulas (58) and (59) the velocity $u(s,t)$ of any cable point s at any time t and the stress σ in this point are connected with the function ϕ' by

$$\frac{u}{c} = \phi'(s+ct) + \phi'(s-ct), \tag{97}$$

$$\frac{\sigma}{E} = \phi'(s+ct) - \phi'(s-ct). \tag{98}$$

Now for $s = 0$ the velocity

$$\dot{x} = u(t)$$

is prescribed and the equation of motion of the mass m is

$$m \cdot \ddot{x} = q\,\sigma_{s=0}.$$

For $s = 0$ the equations (97) and (98), therefore, take the form

$$\phi'(ct) + \phi'(-ct) = \frac{u}{c},$$

$$\phi'(ct) - \phi'(-ct) = \frac{m}{Eq}\,\dot{u}.$$

Setting

$$ct = \xi$$

we obtain by adding and subtracting these equations

$$\phi'(\xi) = \frac{1}{2}\left(\frac{u}{c} + \frac{m}{Eq}\,\dot{u}\right), \tag{99}$$

$$\phi'(-\xi) = \frac{1}{2}\left(\frac{u}{c} - \frac{m}{Eq}\,\dot{u}\right). \tag{100}$$

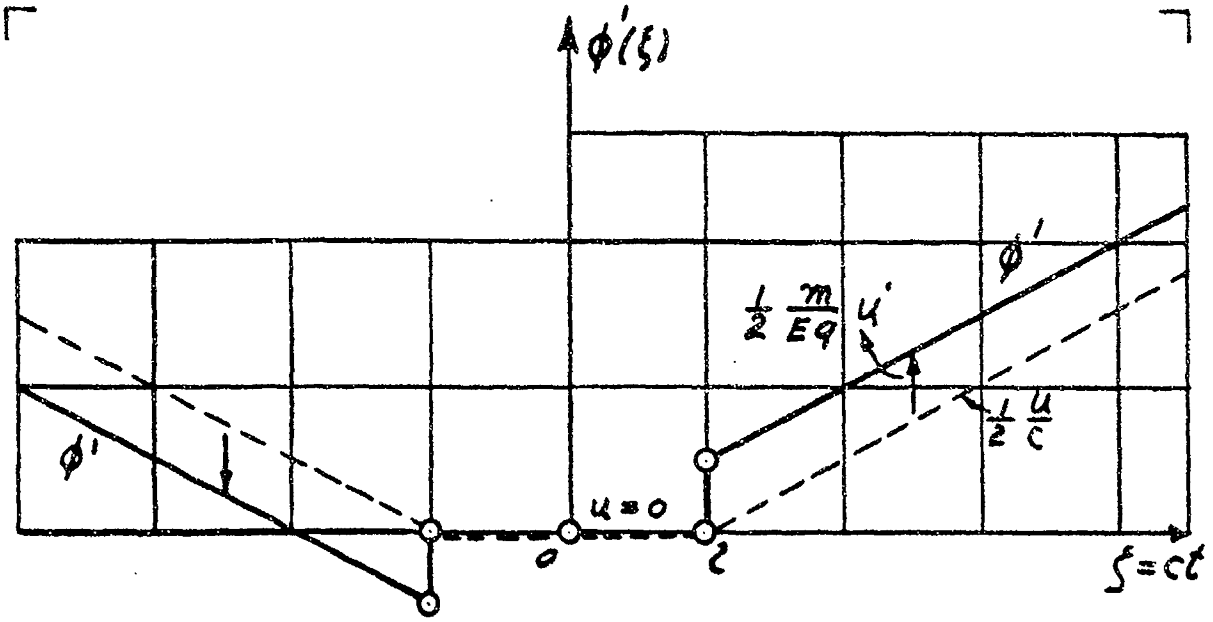


FIGURE 20a

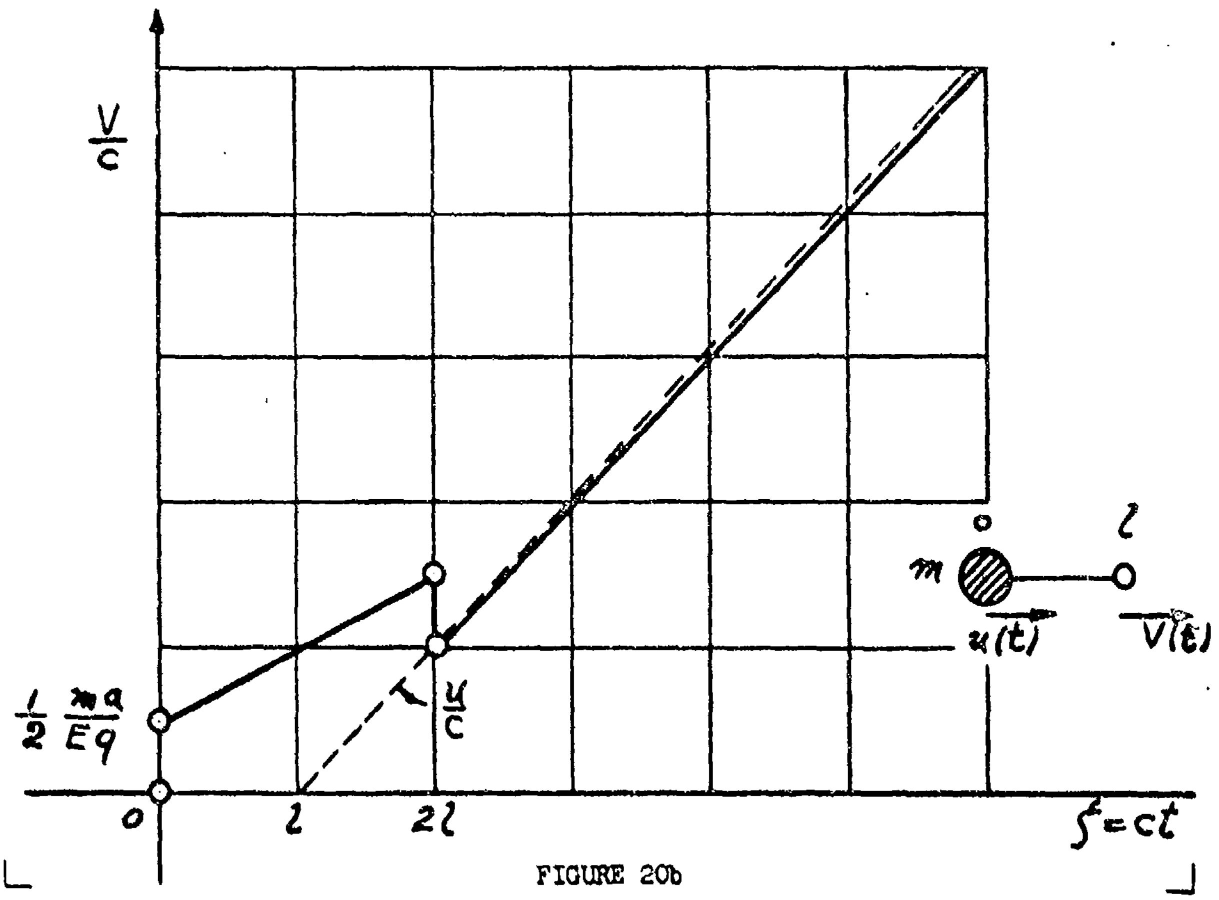


FIGURE 20b

Thus the function ϕ' is determined for all positive and negative values of its argument by the given function $u(t)$ and its derivative $\dot{u}$. However, if the motion of the cable end point is supposed to begin at the time $t=0$ the values of $u(t)$ have to be prescribed equal zero during the time t for which $0 < ct < l$ because the stress wave needs the time $\frac{l}{c}$ to travel from $s=l$ to $s=0$. After the time $\frac{l}{c}$ the values of $u(t)$ can be prescribed arbitrarily. The unknown function $v(t)$ is then determined by the relation

$$\frac{v}{c} = \phi'(l+\zeta) + \phi'(l-\zeta) \tag{101}$$

following from formula (97) for $s=l$ and $ct=\zeta$.

We consider at first the example where a mass has to be moved with a constant acceleration. In this case

$$u = 0 \qquad \text{for} \qquad 0 < \zeta = ct < l$$
$$u = a\left(t - \frac{l}{c}\right) \qquad \text{for} \qquad \zeta = ct > l$$

a being the prescribed constant acceleration of the mass. Accordingly

$$\dot{u} = 0 \qquad\qquad 0 < \zeta = ct < l$$
$$\dot{u} = a \qquad\qquad \zeta = ct > l$$

In Figure 20a the values of $\frac{1}{2}\frac{u}{c}$ and $\frac{1}{2}\frac{m\dot{u}}{Eq}$ which determine $\phi'(\zeta)$ (see formulas (99) and (100)) are plotted as dotted lines versus ζ. For positive values of ζ these values are added and for negative values of ζ subtracted giving the solid curve ϕ'. According to formula (101) the $\frac{v}{c}$ values are obtained and plotted in Figure 20b.

We see that $\frac{v}{c}$ starts with a finite value and increases linearly with t up to the time $t = \frac{2l}{c}$. Here it jumps back to the prescribed value $\frac{u}{c}$ and coincides further on with this value. The result can be explained in the following way. The sudden motion of the cable end point $s=l$ with the finite velocity $\frac{1}{2}\frac{ma}{Eq}\cdot c$ produces a longitudinal impact

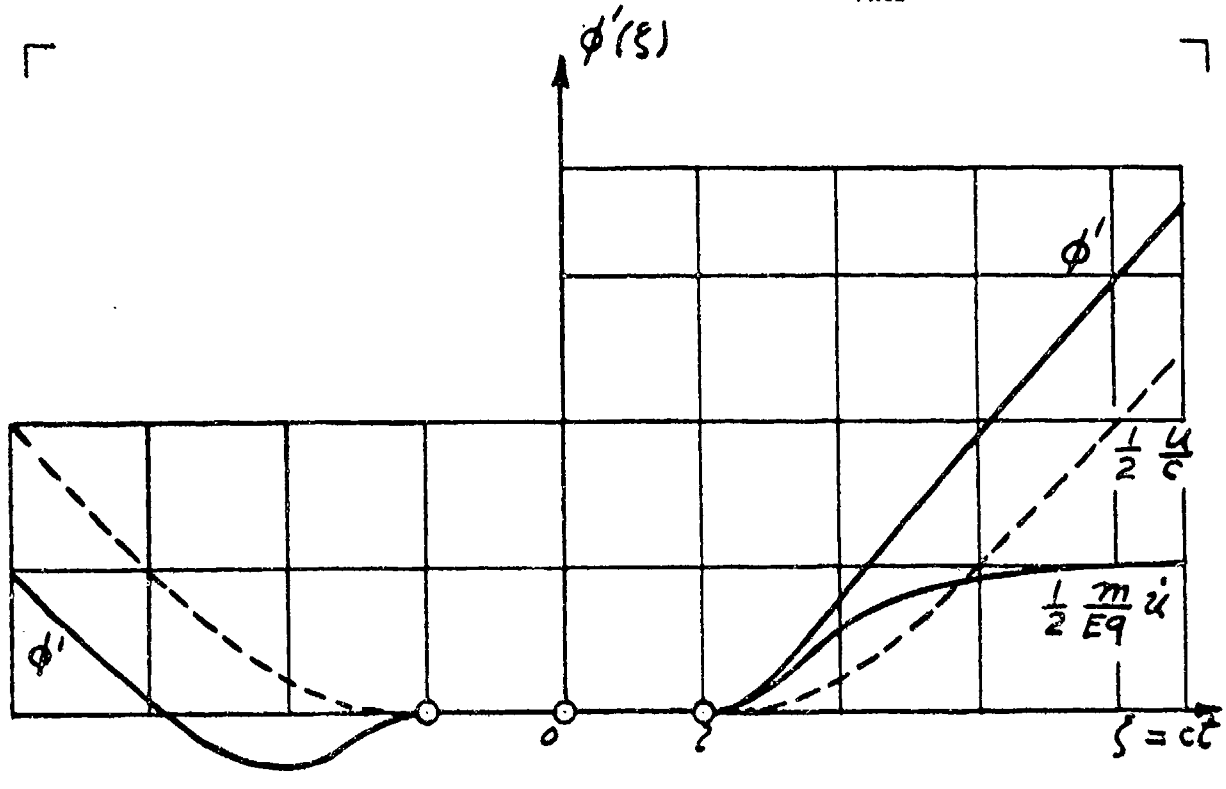


FIGURE 21a

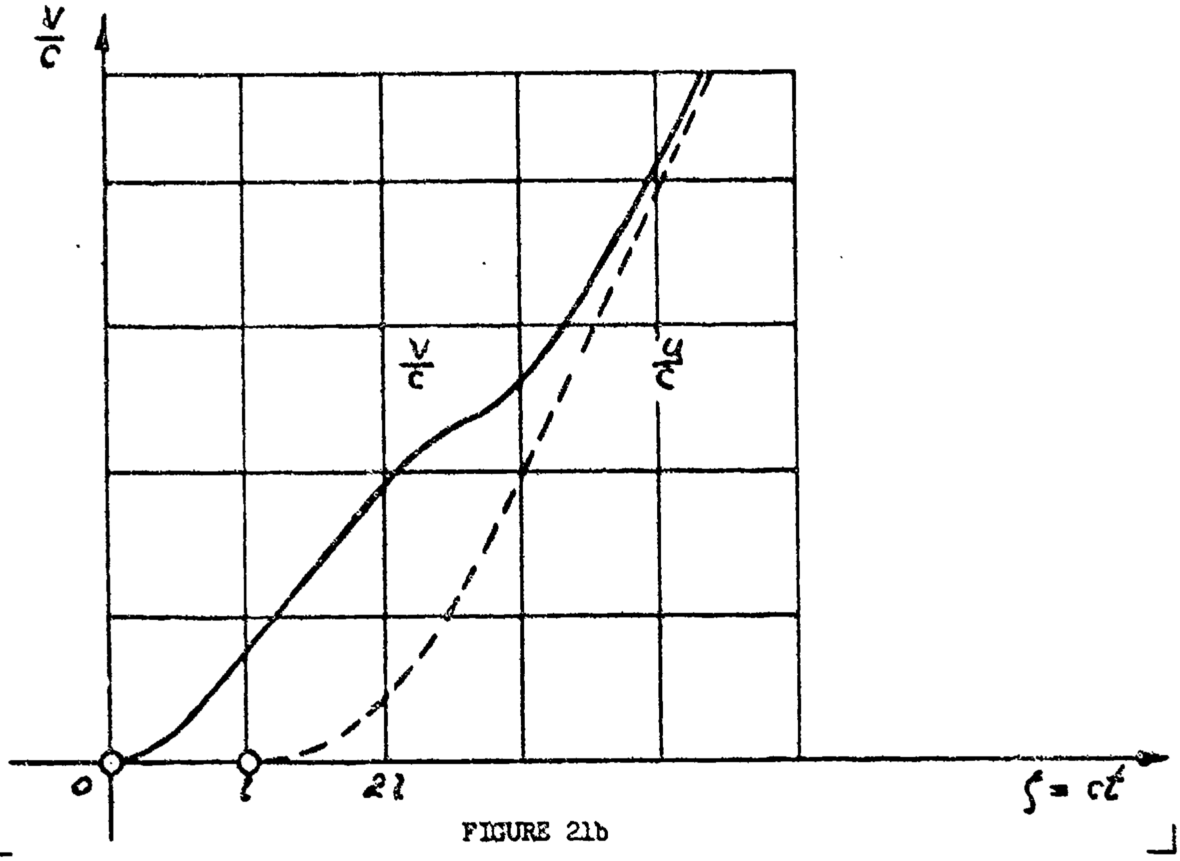


FIGURE 21b

stress which propagates toward the mass m and is completely reflected at it. The reflection runs back toward the end point $\zeta = \ell$ and would be reflected there once more if there would not be a negative impact represented by the jump in the $V-$ curve which cancels the reflection. The impact stress which is completely reflected at the mass m and, therefore, doubled in magnitude, begins to move the mass. As soon as it moves it tends to decrease the stress in the cable, and therewith, the force acting on the mass. Such decrease in stress, however, is prohibited by a properly increasing velocity V. After the discontinuous change of V at the time $t = \frac{2\ell}{c}$ the stress which is equal twice the impact stress remains constant because both cable and points are moving now with the same acceleration.

The general case of the problem can be solved in the same way (see Figure 21). We assume that a desired acceleration $\dot{u}$ of the mass beginning at the time $t = \frac{\ell}{c}$ is prescribed. We plot the curve $\frac{1}{2}\frac{m}{Eq}\dot{u}$ versus $\zeta = ct$, integrate it graphically starting from $\zeta = \ell$ and plot $\frac{1}{2}\frac{u}{c}$. The sum of both curves gives Φ' for $\zeta > 0$. The difference of both curves gives Φ' for $\zeta < 0$ (see Figure 21a). Again formula (101) yields the values $\frac{V}{c}$ (see Figure 21b) which have the curve $\frac{u}{c}$ as an asymptote.

It must be noted that the general problem can be solved in any case graphically without difficulty starting from the graphically given curve $\frac{1}{2}\frac{m}{Eq}\dot{u}$. The practical realization of the solution requires a careful control of the speed V of the cable end point $\zeta = \ell$ and will be easier to obtain, for instance, in the case where the acceleration of the mass approaches a constant value (as shown in Figure 21) than in the case where the constant value is required from the beginning of the motion.

3. Influence of a Mass Between Two Cables on the Stress Propagation

We consider now two in general different cables in straight position which are connected by an inelastic mass m. The two cables are distinguished by the indices 1 and 2 so that, for instance, E_1 is the elasticity modulus of the cable 1. We assume that the two cables are situated along the X axis and that especially X is the coordinate of the mass point m (see Figure 22). We assume further

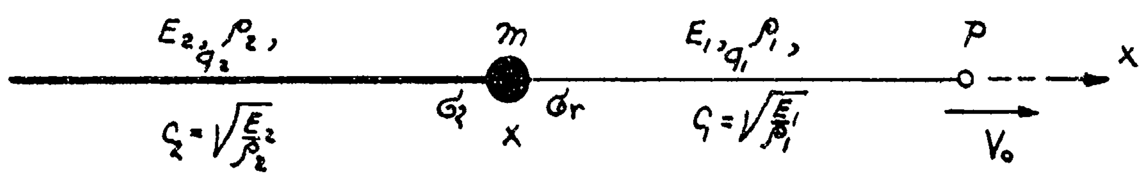


Figure 22

that the system is initially in rest and has a tension T_0. We let move now longitudinally the end point P of the cable 1 with a constant velocity V_0 producing in this way an impact tension which propagates along the cable 1. The problem is to determine in which manner this tension propagates over the mass m and along the cable 2 and how the tension in cable 1 changes during this process. For simplicity we assume that all stress ratios $\frac{\sigma}{E}$ are negligibly small compared with 1. Then, especially, the longitudinal wave velocities in the two cables are

$$c_1 = \sqrt{\frac{E_1}{\rho_1}} \quad , \qquad c_2 = \sqrt{\frac{E_2}{\rho_2}} \; . \tag{102}$$

The initial stresses in the cables 1 and 2 are

$$\sigma_{01} = \frac{T_0}{q_1} \quad , \qquad \sigma_{02} = \frac{T_0}{q_2} .$$

Thus

$$q_1 \sigma_{01} = q_2 \sigma_{02} \tag{103}$$

The stress σ_1, induced at the cable end point P due to the impact with the velocity V_0 is given by

$$\frac{\sigma_1 - \sigma_{01}}{E_1} = \frac{V_0}{c_1} \tag{104}$$

or

$$\sigma_1 = \sigma_{01} + V_0 \sqrt{E_1 \rho_1} . \tag{105}$$

The corresponding induced tension $T_1 = q_1 \sigma_1$. We assume that the stress wave arrives at the mass at the time $t = 0$. The stress difference $\sigma_1 - \sigma_{01}$ is reflected completely at the mass which is at this moment in rest. Immediately after the reflection the stress on the right side of m is, therefore

$$\sigma_{r\,t=0} = \sigma_{01} + 2V_0 \sqrt{E_1 \rho_1} \tag{106}$$

and if m moves with the velocity $\dot{x}$ at the time t

$$\sigma_r = \sigma_{01} + 2V_0 \sqrt{E_1 \rho_1} - \dot{x} \sqrt{E_1 \rho_1} \tag{107}$$

according to formula (51). The stress on the left side of m produced by the velocity $\dot{x}$ is according to the same formula

$$\sigma_l = \sigma_{02} + \dot{x} \sqrt{E_2 \rho_2} . \tag{108}$$

The corresponding tensions are

$$T_r = q_1 \sigma_{01} + q_1 \sqrt{E_1 \rho_1} \, (2V_0 - \dot{x}) , \tag{109}$$

$$T_l = q_2 \sigma_{02} + q_2 \sqrt{E_2 \rho_2} \cdot \dot{x} . \tag{110}$$

The equation of motion of the mass m is

$$m \ddot{x} = T_r - T_l$$

or because of (109), (110) and (103)

$$m \ddot{x} = 2V_0 q_1 \sqrt{E_1 \rho_1} - (q_1 \sqrt{E_1 \rho_1} + q_2 \sqrt{E_2 \rho_2}) \dot{x} \tag{111}$$

We denote

$$\boxed{A = \frac{2q_1 \sqrt{E_1 \rho_1}}{m} , \qquad B = \frac{q_1 \sqrt{E_1 \rho_1} + q_2 \sqrt{E_2 \rho_2}}{m}} \tag{112}$$

Then the velocity $V = \dot{x}$ of the mass m satisfies the equation

$$\dot{V} = A V_0 - B V . \tag{113}$$

Integration of this equation under the initial condition $V=0$ for $t=0$ yields

$$\boxed{\frac{V}{V_0} = \frac{A}{B}\left(1 - e^{-Bt}\right)} \tag{114}$$

which determines the velocity V of the mass m. From (107) and (108) follow with this expression for V the values for the stresses on the right and the left side of the mass

$$\sigma_r = \sigma_{01} + (2V_0 - V)\sqrt{E_1 \rho_1} \tag{115}$$

$$\sigma_l = \sigma_{02} + V\sqrt{E_2 \rho_2} \tag{116}$$

The corresponding tensions are

$$T_r = q_1 \sigma_r , \qquad T_l = q_2 \sigma_l$$

and σ_{01}, σ_{02} satisfy the condition (103)

$$q_1 \sigma_{01} = q_2 \sigma_{02} = T_0 .$$

These formulas describe the stress development in the two cables adjacent to the mass m as long as no stress reflection from point P reaches the mass m.

The formulas (115) and (116) can be written also in the dimensionless form

$$\frac{\sigma_r}{E_1} = \frac{\sigma_{01}}{E_1} + \frac{2V_0 - V}{c_1} , \tag{117}$$

$$\frac{\sigma_l}{E_2} = \frac{\sigma_{02}}{E_2} + \frac{V}{c_2} \tag{118}$$

because of (102).

4. Special Cases and Applications

a. Two cables attached to each other by a negligible mass.

If m decreases toward zero the constants A and B become infinite and (114) yields

$$\frac{V}{V_0} = \frac{2q_1\sqrt{E_1\rho_1}}{q_1\sqrt{E_1\rho_1} + q_2\sqrt{E_2\rho_2}} .$$

Introducing V from this equation into the relations (115) and (116) for σ_r and σ_l we obtain

$$\sigma_r = \sigma_{01} + 2v_0\,\frac{q_2\sqrt{E_1\rho_1}\,\sqrt{E_2\rho_2}}{q_1\sqrt{E_1\rho_1} + q_2\sqrt{E_2\rho_2}}$$

and

$$\sigma_l = \sigma_{02} + 2v_0\,\frac{q_1\sqrt{E_1\rho_1}\,\sqrt{E_2\rho_2}}{q_1\sqrt{E_1\rho_1} + q_2\sqrt{E_2\rho_2}} .$$

Thus

$$T_r = T_l .$$

We have now

$$T_r - T_0 = 2v_0\,\frac{q_1\sqrt{E_1\rho_1}\cdot q_2\sqrt{E_2\rho_2}}{q_1\sqrt{E_1\rho_1} + q_2\sqrt{E_2\rho_2}} . \tag{119}$$

The impact tension T_1 produced by the velocity V_0 in the cable 1 is because of (105) given by

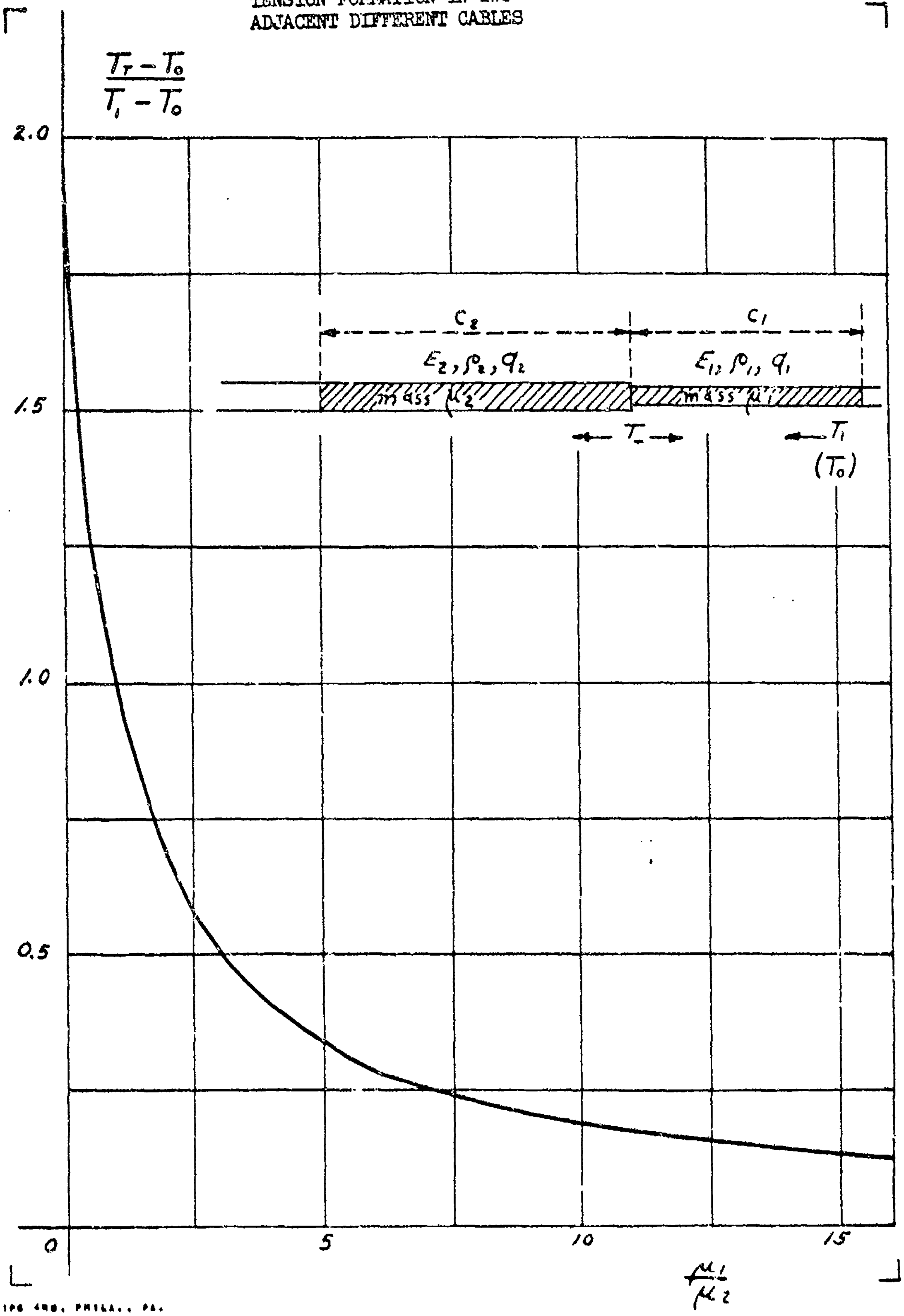
$$T_1 - T_0 = V_0\, q_1\sqrt{E_1\rho_1} . \tag{120}$$

If we divide (119) by (120) we get

$$\boxed{\frac{T_r - T_0}{T_1 - T_0} = \frac{2}{1 + \frac{q_1}{q_2}\sqrt{\frac{E_1\rho_1}{E_2\rho_2}}}} \tag{121}$$

The physical meaning of this formula becomes obvious if we introduce the longitudinal wave velocities C_1 and C_2 instead of the elasticity moduli.

FIGURE 23
TENSION FORMATION IN TWO
ADJACENT DIFFERENT CABLES



Then

$$\frac{T_r - T_0}{T_1 - T_0} = \frac{2}{1 + \frac{q_1 \rho_1 c_1}{q_2 \rho_2 c_2}} .$$

Now

$$\mu_1 = q_1 \rho_1 c_1 , \qquad \mu_2 = q_2 \rho_2 c_2$$

are the masses of the two cables through which any tension wave propagates per second so that formula (121) can be written also in the form

$$\boxed{\frac{T_r - T_0}{T_1 - T_0} = \frac{2}{1 + \frac{\mu_1}{\mu_2}}} \tag{122}$$

Figure 23 represent[illegible] [illegible]is result graphically.

The formula shows especially that there is no change of the tension if a tension wave propagates over the connection of the two cables if and only if

$$\mu_1 = \mu_2$$

For two cables of the same material $(E_1 = E_2 \; , \; \rho_1 = \rho_2)$ formula (121) takes the form

$$\frac{T_r - T_0}{T_1 - T_0} = \frac{2}{1 + \frac{q_1}{q_2}} \tag{123}$$

which shows that the tension increases if it propagates from one cable to a second cable of larger cross section area.

b. Mass between two equal cables

We apply the general theory of section III 3 to the case where

$$E_1 = E_2 = E , \quad \rho_1 = \rho_2 = \rho , \quad q_1 = q_2 = q .$$

Then

$$A = B = \frac{2q\sqrt{E\rho}}{m} \tag{124}$$

and

$$\frac{v}{V_0} = 1 - e^{-Bt}$$

according to the formulas (112) and (114). Since now $\sigma_{01} = \sigma_{02} = \sigma_0$ the relations (115) and (116) for σ_r and σ_ℓ yield

$$\frac{\sigma_r - \sigma_0}{\sigma_1 - \sigma_0} = 1 + e^{-Bt}, \tag{125}$$

$$\frac{\sigma_l - \sigma_0}{\sigma_1 - \sigma_0} = 1 - e^{-Bt}. \tag{126}$$

Figure 24 represents this result graphically.

The incoming stress is always completely reflected at the mass m. This case is not quite realistic because no mass is actually inelastic. In spite of this fact, it shall be used here for the determination of the influence of a sheave on the stress propagation. We replace the mass of the sheave for this purpose by the equivalent mass on its rim and consider this mass attached to the cable. For a Mark 5 arresting gear sheave, for instance, this equivalent mass is

$$m = 1 \quad lbs \cdot sec^2/ft.$$

For a 1" diameter cable is according to table 1 of section I 5

$$E = 18.3 \cdot 10^8 \; lbs/ft^2, \qquad \rho = 18.12 \; lbs \; sec^2/ft^4,$$

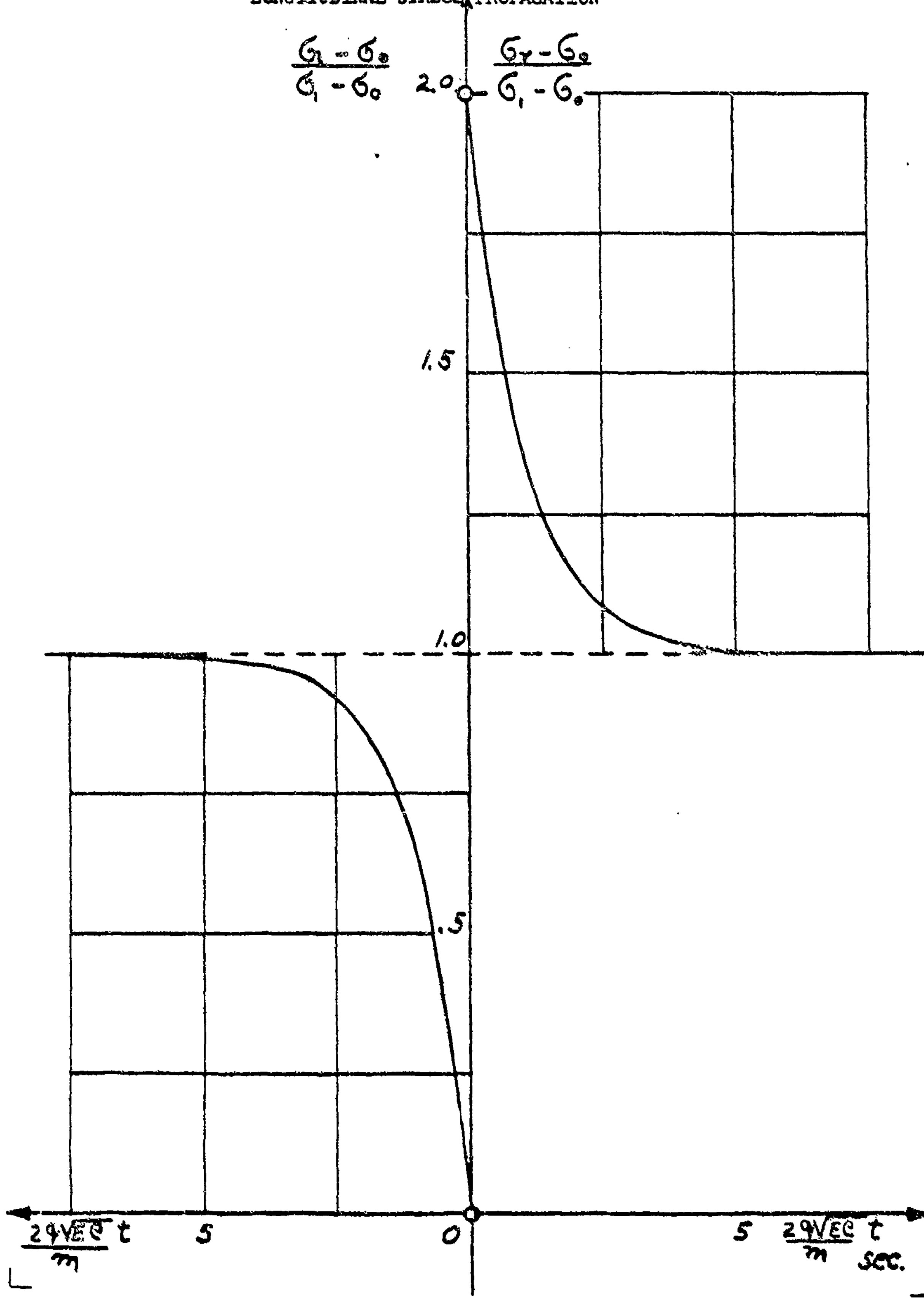
$$q = 0.00274 \; ft^2.$$

The constant

$$B = \frac{2q\sqrt{E\rho}}{m} = 1000$$

in this case. Figure 24 shows the remarkable fact that the sheave turns so fast that after 0.001 seconds only one-third of the accumulated stress is left and that after 0.005 seconds the stresses on both sides of the sheave are practically balanced.

FIGURE 24
INFLUENCE OF AN INELASTIC MASS ON THE
LONGITUDINAL STRESS PROPAGATION



c. Determination of Links for Strain Gage Measurements

If a link is used for a strain gage measurement of the tension in a cable the result of section III 4a should be considered. The link should not obstruct the tension propagation. Therefore, the diameter of the link should be chosen, so that

$$\mu_1 = \mu_2$$

If E_1, ρ_1, q_1 are the cable data and E_2, ρ_2, q_2 the data of the link which now is considered as the second elastic cable or bar then

$$q_1 \sqrt{E_1 \rho_1} = q_2 \sqrt{E_2 \rho_2} \tag{127}$$

is the condition which has to be satisfied. For example, we choose a cable where

$$E_1 = 18.3 \cdot 10^8 \ lbs/ft^2, \quad \rho_1 = 18.12 \ lbs \cdot sec^2/ft^4.$$

If the link is a steel bar where

$$E_2 = 40.7 \cdot 10^8 \ lbs/ft^2, \quad \rho_2 = 15.15 \ lbs \cdot sec^2/ft^4$$

equation (127) yields

$$q_2 = 0.733\, q_1$$

as cross section area of the link q_1 being the metallic cross section area of the cable.

d. Stress Propagation through Three Cables

We apply the result of III 4a to the solution of the following problem: Two cables C_1 and C_2 with infinite length are connected by a third cable or bar C with the length l between C_1 and C_2 (see Figure 25). The system is initially in rest and has the tension T_0. Along C_1 a tension T_1 propagates toward C. The tension within the cable C has to be determined as function of the time t.

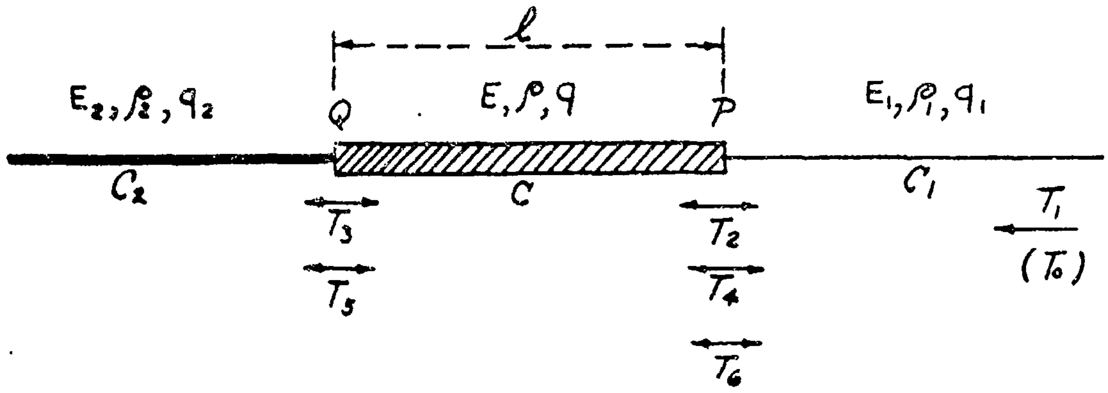


Figure 25

We denote, as before

$$\mu_1 = q_1 \rho_1 c_1 \quad , \qquad \mu_2 = q_2 \rho_2 c_2 \quad , \qquad \mu = q \rho c$$

where

$$c_1 = \sqrt{\frac{E_1}{\rho_1}} \quad , \qquad c_2 = \sqrt{\frac{E_2}{\rho_2}} \quad , \qquad c = \sqrt{\frac{E}{\rho}} \; .$$

If the tension T_1 passes over the connection P between C_1 and C it changes to the value T_2 determined by formula (122) which, applied to the present case, yields

$$\frac{T_2 - T_0}{T_1 - T_0} = \frac{2}{1 + \frac{\mu_1}{\mu}} \; .$$

The tension T_2 propagates along C_1 and C . If it reaches the connection Q between C and C_2 it changes to the tension T_3 which again isdetermined by formula (122) yielding here

$$\frac{T_3 - T_0}{T_2 - T_0} = \frac{2}{1 + \frac{\mu}{\mu_2}} \; .$$

The tension T_3 propagates along C and C_2 . If it reaches point P it changes to T_4 . The initial tension is now T_2 , the incoming tension T_4 . Therefore, formula (122) applies in the form

$$\frac{T_4 - T_2}{T_3 - T_2} = \frac{2}{1 + \frac{\mu}{\mu_1}} \; .$$

T_4 propagates along C_1 and C and so on. Thus

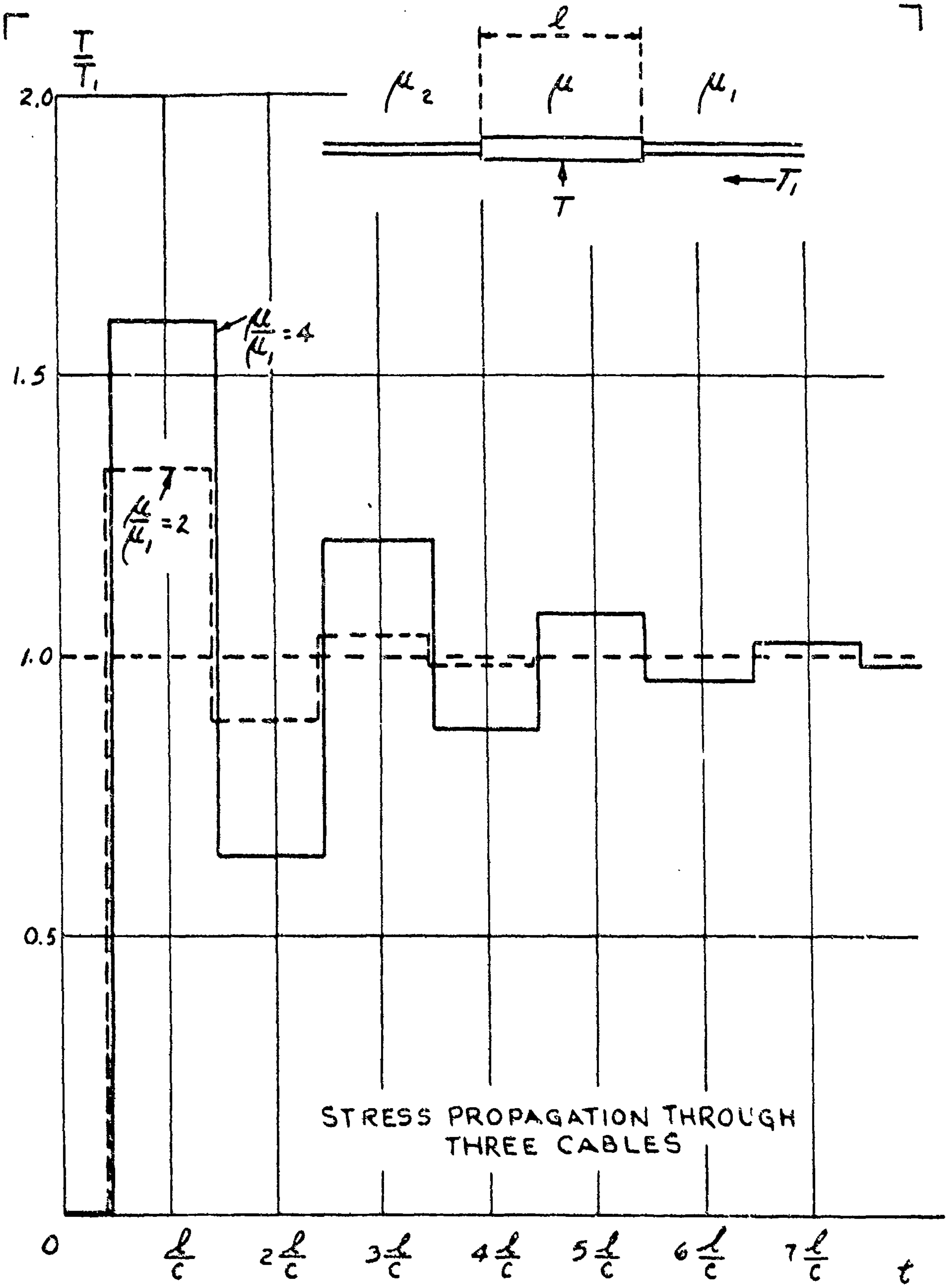

$\frac{T}{T_1}$
2.0
1.5
1.0
0.5
0
μ_2
μ
μ_1
ℓ
T
T_1
$\frac{\mu}{\mu_1} = 4$
$\frac{\mu}{\mu_1} = 2$
$\frac{\ell}{c}$
$2\frac{\ell}{c}$
$3\frac{\ell}{c}$
$4\frac{\ell}{c}$
$5\frac{\ell}{c}$
$6\frac{\ell}{c}$
$7\frac{\ell}{c}$
t
STRESS PROPAGATION THROUGH THREE CABLES

FIGURE 26

$$\frac{T_5' - T_3}{T_4 - T_3} = \frac{2}{1 + \mu/\mu_2}, \qquad \frac{T_6' - T_4}{T_5 - T_4} = \frac{2}{1 + \mu/\mu_1}, \qquad \frac{T_7 - T_5}{T_6 - T_5} = \frac{2}{1 + \mu/\mu_2},$$

and so on. If T_0, T_1 and the ratios μ/μ_1 and μ/μ_2 are given, these formulas determine successively T_2, T_3, ... The time for a tension wave needed to propagate from P to Q is equal $\frac{l}{c}$. Thus the tension at any point of C at any time can be computed.

Figure 26 shows the result of the computation for $T_0 = 0$ and the values

$$\mu/\mu_1 = \mu/\mu_2 = 2$$

and

$$\mu/\mu_1 = \mu/\mu_2 = 4$$

for the center of the cable C. If C is a short elastic link this figure proves that the increase in tension is damped out in a very short time since $\frac{l}{c}$ is a very small quantity.

If, for instance, the cables C_1 and C_2 are 1" diameter cables, as used before, so that

$$E_1 = E_2 = 18.3 \cdot 10^6 \ lbs/ft^2, \qquad \rho_1 = \rho_2 = 18.12 \ lbs. sec^2/ft^4,$$
$$q_1 = q_2 = 0.00274 \ ft^2$$

and the link is a steel bar with

$$E = 40.7 \cdot 10^8 \ lbs/ft^2, \qquad \rho = 15.15 \ lbs. sec^2/ft^4,$$
$$l = 1 \ ft, \qquad q = 0.00545 \ ft^2$$

which has the same diameter as the cables we get

$$\mu/\mu_1 = \mu/\mu_2 = \frac{q}{q_1}\sqrt{\frac{E\rho}{E_1\rho_1}} = 2.72$$

and

$$\frac{l}{c} = 0.0001 \ sec.$$

The maximum tension is in this case given by

$$T_{max}/T_1 = 1.46.$$

This tension is damped out to the value T_1 in about 0.0005 seconds. However, it must be taken into consideration that this maximum tension actually occurs and not only in the link but also in the cables attached to it.

CHAPTER II: TWO-DIMENSIONAL MOTION OF A CABLE

1. Mathematical Description of the Two-Dimensional Motion.

We assume that a cable is situated initially at the time $t=0$ along the x-axis of a rectangular x, y coordinate system (see Figure 27), and that it has at this time a constant initial stress σ_0.

The cable moves with increasing time t out of this position within the x, y - plane. A cable point with the initial x-coordinate s and the initial y coordinate $y = 0$ has at the time t a position x, y within this plane.

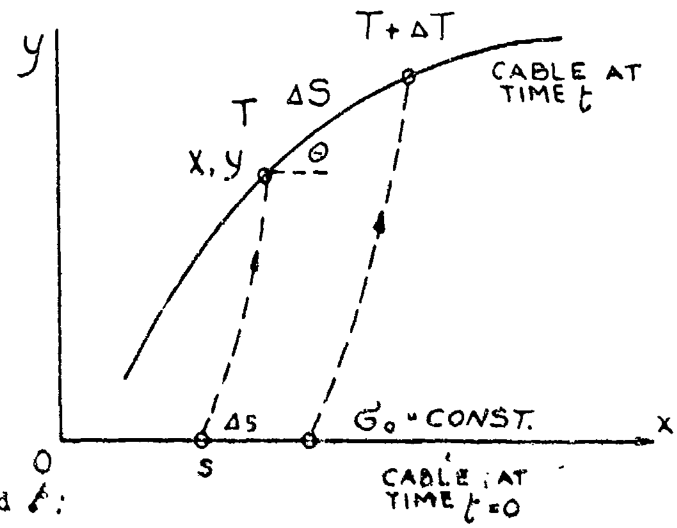


FIGURE 27

x and y are functions of s and t:

$$x = f(s,t), \quad y = g(s,t) \tag{128}$$

If we keep s constant, these two functions of t are the equations of the curve which the fixed cable point s describes during the motion of the cable. If we keep t constant they are the equations of the curve along which the cable is situated at the fixed time t. The variable parameter s corresponds to the cable points in this position. The two functions (128) are not completely arbitrary functions. They have to satisfy the conditions

$$f(s, 0) = s, \quad g(s, 0) = 0. \tag{129}$$

During the motion of the cable, it is generally distorted so that the distance Δs between to neighbor points s and $s + \Delta s$ in initial position changes to a distance ΔS at the time t resulting in a tension T in the cable at this point which will be generally different from the initial tension T_o. The tension T is in this way a function of s and t too and so is the angle Θ between the cable element ΔS and the x-axis. This angle is defined as the angle about which the positive x-direction has to be turned counterclockwise in order to coincide with the direction of the cable element which corresponds to increasing s values.

Again we assume that the cross section area q of the cable does not change with the motion. The stress σ in the cable point corresponding to s at the time t is then defined by

$$\sigma = \frac{T}{q} \tag{130}$$

and is a function of s and t too. The density however changes now with σ. We denote by ρ the density of the cable under its initial stress σ_o = constant and by ρ_o the density at zero stress.

2. Equations of Two-Dimensional Motion and Stress

The equations $x = f(s,t)$, $y = g(s,t)$ which describe the motion of a cable are completely arbitrary functions except that they have to satisfy the condition (129) if we admit arbitrary forces acting from outside on the cable elements. If we suppose, however, that a cable element moves under the influence of the tension within the cable only the functions $x = f(s,t)$ and $y = g(s,t)$ have to satisfy certain conditions to be derived in the following.

For this purpose, we consider the element ΔS of the cable at the fixed time t corresponding to the element Δs at the time $t = 0$ situated between the points s and $s + \Delta s$ (see Figure 27) on the X-axis. The components of the tension T in the cable point corresponding to s at the time t are

$$T\cos\theta \quad , \qquad T\sin\theta$$

in the x- and the y- direction. In the cable point corresponding to $s + \Delta s$ at the same time t these components are

$$T\cos\theta + \frac{\partial}{\partial s}(T\cos\theta)\Delta s \quad , \qquad T\sin\theta + \frac{\partial}{\partial s}(T\sin\theta)\Delta s.$$

The components of the force which moves the element ΔS are therefore

$$\frac{\partial}{\partial s}(T\cos\theta)\Delta s \quad , \qquad \frac{\partial}{\partial s}(T\sin\theta)\Delta s$$

The mass of the element ΔS is equal to the mass of the element Δs which is

$$\rho q \Delta s$$

According to Newton's law, therefore

$$\rho q \Delta s \cdot \frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial s}(T\cos\theta)\Delta s,$$

$$\rho q \Delta s \cdot \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s}(T\sin\theta)\Delta s,$$

which can be written in the form

$$\boxed{\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{\sigma}{\rho}\cos\theta\right), \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{\sigma}{\rho}\sin\theta\right)} \tag{131}$$

These are two conditions for the four unknown functions x, y, σ, θ of the variables s and t. We obtain another two conditions from the application of Hooke's law.

The elongation of the element ΔS with respect to the initial element Δs having the stress σ_0 is equal $\Delta S - \Delta s$. Thus, Hooke's law (formula (6)) yields

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \lim_{\Delta s \to 0}\left(\frac{\Delta S}{\Delta s} - 1\right)$$

or

$$\frac{1 + \frac{\sigma}{E}}{1 + \frac{\sigma_0}{E}} = \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} = \frac{dS}{ds}. \tag{132}$$

On the other hand we have

$$\frac{\partial x}{\partial s} = \frac{dS}{ds}\cos\theta, \qquad \frac{\partial y}{\partial s} = \frac{dS}{ds}\sin\theta \tag{133}$$

Therefore

$$\boxed{\frac{\partial x}{\partial s} = \frac{1 + \frac{\sigma}{E}}{1 + \frac{\sigma_0}{E}}\cos\theta, \quad \frac{\partial y}{\partial s} = \frac{1 + \frac{\sigma}{E}}{1 + \frac{\sigma_0}{E}}\sin\theta} \tag{134}$$

Thus we have the result:

<u>The equations (131) and (132) are the conditions which the functions $x(s,t)$, $y(s,t)$, $\sigma(s,t)$ and $\theta(s,t)$ describing the motion, the stress and the slope of the cable have to satisfy if the motion occurs under the influence of the cable tension only.</u> *)

Eliminating x and y from these four equations we obtain for σ and θ alone the relations

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}\cos\theta\right) = \frac{\partial^2}{\partial s^2}\left(\frac{\sigma}{\rho}\cos\theta\right), \quad \frac{\partial^2}{\partial t^2}\left(\frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}\sin\theta\right) = \frac{\partial^2}{\partial s^2}\left(\frac{\sigma}{\rho}\sin\theta\right)} \tag{135}$$

* Compare references 6, 7, 8 and 11

(135)

If we use instead the coordinates of x and y for the description of the cable motion the velocity components of the cable point

$$u = \frac{\partial x}{\partial t} \quad , \qquad v = \frac{\partial y}{\partial t} \tag{136}$$

the equations (131) and (134) take the form *

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(\frac{\sigma}{\rho}\cos\theta\right) , \qquad \frac{\partial v}{\partial t} = \frac{\partial}{\partial s}\left(\frac{\sigma}{\rho}\sin\theta\right) , \tag{137}$$

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}\cos\theta\right) , \qquad \frac{\partial v}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}\sin\theta\right) . \tag{138}$$

It should be noted that all these differential equations are non-linear equations. Therefore exact solutions cannot generally be superposed linearly in order to obtain other exact solutions.

For small values of $\frac{\sigma}{E}$ and $\frac{\sigma_0}{E}$ compared with 1 the expression $1+\frac{\sigma}{E} / 1+\frac{\sigma_0}{E}$ in the preceding formulas can be replaced by

$$\frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}} = 1 + \frac{\sigma-\sigma_0}{E} \tag{139}$$

The formulas resulting in this way have been used by other authors. ** There is, however, no essential simplification in using this approximation. ***

* Compare reference 12 ** Reference 23 *** For other approximations, compare references 21 and 22

3. Wave Velocities

If a cable element moves parallel to itself so that θ is constant during its motion both equations (135) yield for the stress σ the same condition

$$\frac{\partial^2 \sigma}{\partial t^2} = c^2 \frac{\partial^2 \sigma}{\partial s^2} \tag{140}$$

where

$$\boxed{c^2 = \left(1 + \frac{\sigma_0}{E}\right)\frac{E}{\rho}} \tag{141}$$

Equation (140) is the classical wave equation for the function $\sigma(s, t)$ which has been discussed in Sections II 2 and 3 where instead of σ the variable X occurred. The results of those sections applied to σ show that in the present case the stress value σ propagates with respect to s with the velocity c given by formula (141) c is called the longitudinal wave velocity. For $\sigma_0 = 0$ this velocity is the same as in the classical theory of the vibrating string.

If we consider further a cable element which has during its motion a constant stress σ then the equations (135) can be combined by multiplying the first one by $\cos \theta_0$, the second one by $\sin \theta_0$ and subtracting both equations. This yields

$$\frac{\partial^2}{\partial t^2} \cos(\theta - \theta_0) = \bar{c}^2 \frac{\partial^2}{\partial s^2} \cos(\theta - \theta_0) . \tag{142}$$

Here θ_s is an arbitrary constant and

$$\boxed{\bar{c}^2 = \frac{1 + \frac{\sigma_0}{E}}{1 + \frac{\sigma}{E}} \cdot \frac{E}{\rho}} \tag{143}$$

Again equation (142) is the classical wave equation, now for the quantity $\cos(\theta - \theta_s)$ as function of s and t and $\bar{c}$ is the velocity with which θ propagates with respect to s. We call $\bar{c}$ the transverse wave velocity. It coincides with the transverse wave velocity $\sqrt{\frac{\sigma}{\rho}}$ of the classical theory of the vibrating string if $\frac{\sigma}{E}$ and $\frac{\sigma_0}{E}$ are negligibly small.

[illegible]

If both values are small but not negligibly small compared with unity formula (143) yields approximately

$$\bar{c}^2 = \frac{\frac{\sigma}{\varrho}}{1 + \frac{\sigma - \sigma_0}{E}} . \tag{144}$$

From the formulas (141) and (143) the following general relation is obtained

$$\boxed{\frac{\bar{c}}{c} = \sqrt{\frac{\frac{\sigma}{E}}{1 + \frac{\sigma}{E}}}} \tag{145}$$

We consider additionally the case of a cable motion for which the stress σ and the angle θ are constants. The stress has to be then equal to the initial stress σ_0 and the angle $\theta = 0$ in accordance with the initial conditions. From the basic equations (131) follows in this case that

$$\frac{\partial^2 x}{\partial t^2} = 0, \qquad \frac{\partial^2 y}{\partial t^2} = 0 \tag{146}$$

and from the basic equations (134) that

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial y}{\partial s} = 0 . \tag{147}$$

The conditions (146) yield

$$x = f_0(s) + f_1(s)\, t, \qquad y = g_0(s) + g_1(s)\, t$$

where the functions of s are arbitrary functions. If we substitute these values in (147) we obtain

$$f_0'(s) + f_1'(s)\, t = 1$$

$$g_0'(s) + g_1'(s)\, t = 0$$

for all values of s and t. Thus

$$f_0'(s) = 1 , \qquad f_1'(s) = 0$$

$$g_0'(s) = 0 , \qquad g_1'(s) = 0$$

Therefore

$$f_0(s) = s + a_0 , \qquad f_1(s) = a_1 ,$$

$$g_0(s) = b_0 , \qquad g_1(s) = b_1$$

where a_0, a_1, b_0, b_1 are constants and the motion of the cable is described by the functions

$$x = s + a_0 + a_1 t, \qquad y = b_0 + b_1 t.$$

Because of the initial condition $x = s$ and $y = 0$ for $t = 0$ ~~the constants~~ a_0 and b_0 must be zero. Thus

$$x = s + a_1 t, \qquad y = b_1 t.$$

The motion of the cable is a motion parallel to its initial position with a constant velocity whose components parallel to the $x-$ and the $y-$ axis are respectively

$$\frac{\partial x}{\partial t} = a_1, \qquad \frac{\partial y}{\partial t} = b_1.$$

4. Motion of Singularities

Many applications of a cable especially those where a cable is exposed to impacts lead to the formation of kinks in the cable during its motion. At the point of a kink, the preceding results are not applicable because the functions involved, for instance $\theta(s, t)$, have no derivatives in such point. The derived equations are valid so far only as the required derivatives exist and are continuous. Moreover, we will consider in the following the motion of a cable frequently by replacing approximately its actual shape by a piecewise straight cable. Along the straight parts our previous results will be valid. However, we have to find the relations which are valid if we pass from one straight cable segment to the next one over a singularity as for instance represented by an angle between the two segments.*

For this purpose we assume that a moving cable forms an angle with straight legs at a point P which is moving parallel to itself with a constant velocity while the cable passes through the point P (see Figure 28). We assume further that we are moving with the point P so that the angle is in rest with respect to our reference system. The angles θ_1 and θ_2 and the stresses σ_1 and σ_2 in the two legs are constant but the cable is moving longitudinally through the point P say with the velocities u_1 and u_2 in the two legs respectively. The cross section area q of the cable is supposed to be constant. If the stresses σ_1 and σ_2 would be different the densities ρ_1 and ρ_2 would be different accordingly.

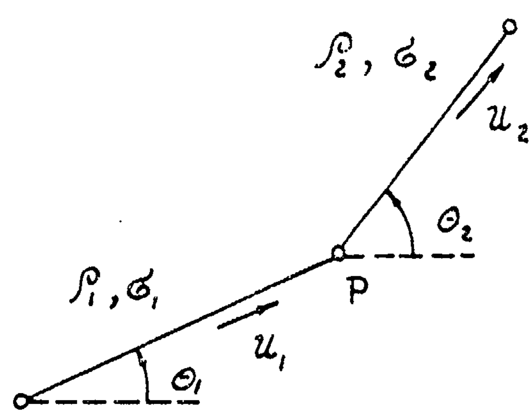


FIGURE 28

* Compare Reference 12.

The mass per second which passes through a marked point on one side of P must be the same as the mass per second passing through a marked point on the other side of P. Thus we have the continuity equation

$$\rho_1 u_1 = \rho_2 u_2 \tag{148}$$

for the mass flow of the cable through the singularity P.

A second relation follows from Hooke's law (formula (6)). Considering σ_1 as initial stress, we have

$$\frac{\frac{\sigma_2 - \sigma_1}{E}}{1 + \frac{\sigma_1}{E}} = \frac{u_2 - u_1}{u_1} \tag{149}$$

because $(u_2 - u_1)\Delta t$ ... is the elongation which an element of the length $u_1 \Delta t$ experiences if it passes over the point P.

Finally two more relations are obtained from momentum considerations. The momentum equation for the direction u_1 yields

$$\sigma_2 \cos(\theta_2 - \theta_1) - \sigma_1 = \rho_2 u_2 \cdot u_2 \cos(\theta_2 - \theta_1) - \rho_1 u_1 \cdot u_1$$

and in the direction perpendicular to u_1

$$\sigma_2 \sin(\theta_2 - \theta_1) = \rho_2 u_2 \cdot u_2 \sin(\theta_2 - \theta_1).$$

Therefore, respectively

$$(\sigma_2 - \rho_2 u_2^2)\cos(\theta_2 - \theta_1) = \sigma_1 - \rho_1 u_1^2 \tag{150}$$

and

$$(\sigma_2 - \rho_2 u_2^2)\sin(\theta_2 - \theta_1) = 0. \tag{151}$$

From equation (151) follows now that one of the three cases

(a) $\sigma_2 - \rho_2 u_2^2 = 0$, (b) $\theta_2 - \theta_1 = 0$, (c) $\theta_2 - \theta_1 = \pi$

holds.

Case (a): Because of

$$\sigma_2 - \rho_2 u_2^2 = 0$$

and equation (150) also

$$\sigma_1 - \rho_1 u_1^2 = 0.$$

Thus

$$\sigma_2 - \sigma_1 = \rho_2 u_2^2 - \rho_1 u_1^2$$

or because of (148)

$$\sigma_2 - \sigma_1 = \rho_1 u_1 u_2 - \rho_1 u_1^2 = \rho_1 u_1 (u_2 - u_1)$$

or

$$\frac{\sigma_2 - \sigma_1}{\sigma_1} = \frac{u_2 - u_1}{u_1} . \tag{152}$$

So equation (149) takes the form

$$\frac{\frac{\sigma_2 - \sigma_1}{E}}{1 + \frac{\sigma_1}{E}} = \frac{\frac{\sigma_2 - \sigma_1}{E}}{\frac{\sigma_1}{E}}$$

which yields

$$\sigma_2 = \sigma_1$$

Thus from equation (152)

$$u_2 = u_1$$

and from (148)

$$\rho_2 = \rho_1 .$$

Dynamic equilibrium exists, then if

$$\sigma_1 = \rho_1 u_1^2 .$$

Case (b): If $\Theta_2 = \Theta_1$ equation (150 yields

$$\sigma_2 - \rho_2 u_2^2 = \sigma_1 - \rho_1 u_1^2$$

or

$$\sigma_2 - \sigma_1 = \rho_2 u_2^2 - \rho_1 u_1^2 = \rho_1 u_1 (u_2 - u_1)$$

because of (148). Therefore equation (149) takes the form

$$\frac{\frac{\sigma_2 - \sigma_1}{E}}{1 + \frac{\sigma_1}{E}} = \frac{\frac{\sigma_2 - \sigma_1}{E}}{\frac{\rho_1 u_1^2}{E}} .$$

Thus again $\sigma_2 = \sigma_1$, $u_2 = u_1$, $\rho_2 = \rho_1$ as before or if $\sigma_2 \neq \sigma_1$

$$u_1 = \sqrt{\left(1+\frac{\sigma_1}{E}\right)\frac{E}{\rho_1}}$$

which is the longitudinal wave velocity (see formula (141)). Correspondingly follows in this case

$$u_2 = \sqrt{\left(1+\frac{\sigma_2}{E}\right)\frac{E}{\rho_2}}$$

which is again the longitudinal wave velocity.

Case (c): If $\theta_2 - \theta_1 = \pi$ formula (150) yields

$$\sigma_2 - \rho_2 u_2^2 = -(\sigma_1 - \rho_1 u_1^2).$$

If we introduce the density ρ_0 for zero stress by

$$\frac{\rho_0}{\rho_1} = 1+\frac{\sigma_1}{E}, \qquad \frac{\rho_0}{\rho_2} = 1+\frac{\sigma_2}{E}$$

according to formula (15) and the longitudinal wave velocity

$$c_0 = \sqrt{\frac{E}{\rho_0}}$$

this equation can be written in the form

$$\frac{\left(\frac{u_1}{c_0}\right)^2}{1+\frac{\sigma_1}{E}} + \frac{\left(\frac{u_2}{c_0}\right)^2}{1+\frac{\sigma_2}{E}} = \frac{\sigma_1}{E} + \frac{\sigma_2}{E}. \tag{153}$$

On the other hand equation (149) can be written in the form

$$\frac{\frac{u_1}{c_0}}{\frac{u_2}{c_0}} = \frac{1+\frac{\sigma_1}{E}}{1+\frac{\sigma_2}{E}}. \tag{154}$$

Both equations (153) and (154) determine the velocities u_1 and u_2 as functions of σ_1 and σ_2 by the intersection of the ellipse (153) with the straight line (154).

5. Oblique, Transverse and Reverse Impact

The endpoint O of a straight long cable with the initial stress σ_0 is assumed to move suddenly with a constant velocity V_0 in a given direction β (see Figure 29). The cable moves than at any time after the impact in two straight parts, the part PQ in the direction β with the velocity V_0 and the part RQ in the direction from R to Q while the part beyond R is at this moment at rest.

This can be derived by elementary physical considerations. The moving force produces longitudinal and transverse displacements of the cable particles with the possibility that the longitudinal displacements propagate with a different velocity along the cable from that of the transverse displacements. At the same time a particle at Q obtains a displacement in the direction β while a particle at R obtains a longitudinal displacement in the direction toward Q. After these sudden displacements no force acts on the two particles anymore. They are moving now with constant speeds. From the homogenity of the material and the constancy of the cross section, it follows that the momentum magnitudes at points Q and R must be at any time the same. Thus the stresses σ_1 along PQ and σ_2 along QR must be constant and it must be $\sigma_1 = \sigma_2$ according to the results of the preceding section if the angle θ satisfies the condition $0 < \theta < \pi$.

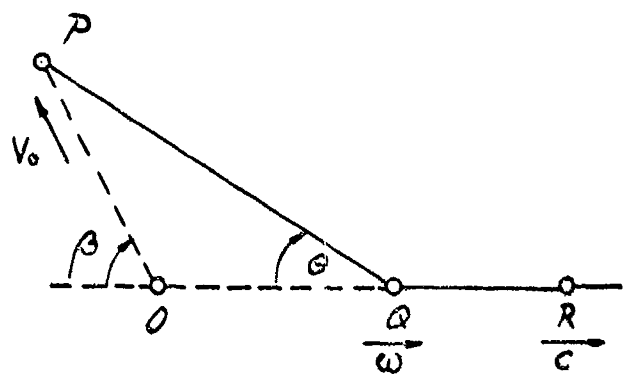


FIGURE 29

From this knowledge we can derive now the formula for the stress due to oblique impact.* For this purpose, Figure 30 shows the cable in its straight initial shape and above it its shape at any time $t > 0$. The endpoint O of the cable moves during the time t in the direction OP with the velocity v_0 from O to P. Thus $OP = v_0 t$. Since the stress along PQR is constant, Q moves with the transverse wave velocity $\bar{c}$ with respect to s according to the result of Section IV 3. If $\bar{Q}$ is the point which corresponds to Q at the time $t = 0$ then $O\bar{Q} = \bar{c}t$. The point $\bar{Q}$ moves toward Q and coincides with Q at the time t. But $\bar{Q}$ starts its motion toward Q only after the longitudinal wave has arrived at $\bar{Q}$ which is the time $\frac{\bar{c}}{c}t$. If t' is the time which $\bar{Q}$ needs to travel up to Q then the longitudinal wave propagates in the same time t' from $\bar{Q}$ to R in the distance ct from O. Thus

$$t' = \left(1 - \frac{\bar{c}}{c}\right)t. \tag{155}$$

In the part QR the cable has a stress σ which is produced by the oblique impact. But here the cable is insensitive to whether the stress has been produced by an oblique or a longitudinal impact with a certain velocity u. Therefore, the cable must move between Q and R with a velocity u which is determined by the longitudinal impact formula (35) so that

$$\frac{\frac{\sigma - \sigma_0}{E}}{1 + \frac{\sigma_0}{E}} = \frac{u}{c}. \tag{156}$$

Now $Q\bar{Q} = ut'$ and therefore

$$\omega t = \bar{c}t - ut' = \bar{c}t - u\left(1 - \frac{\bar{c}}{c}\right)t$$

or

$$\frac{\omega}{c} = \frac{\bar{c}}{c} - \frac{u}{c}\left(1 - \frac{\bar{c}}{c}\right). \tag{157}$$

This relation determines the velocity ω of the kink Q if the stress is known.

* Compare the approximate results. Reference 8

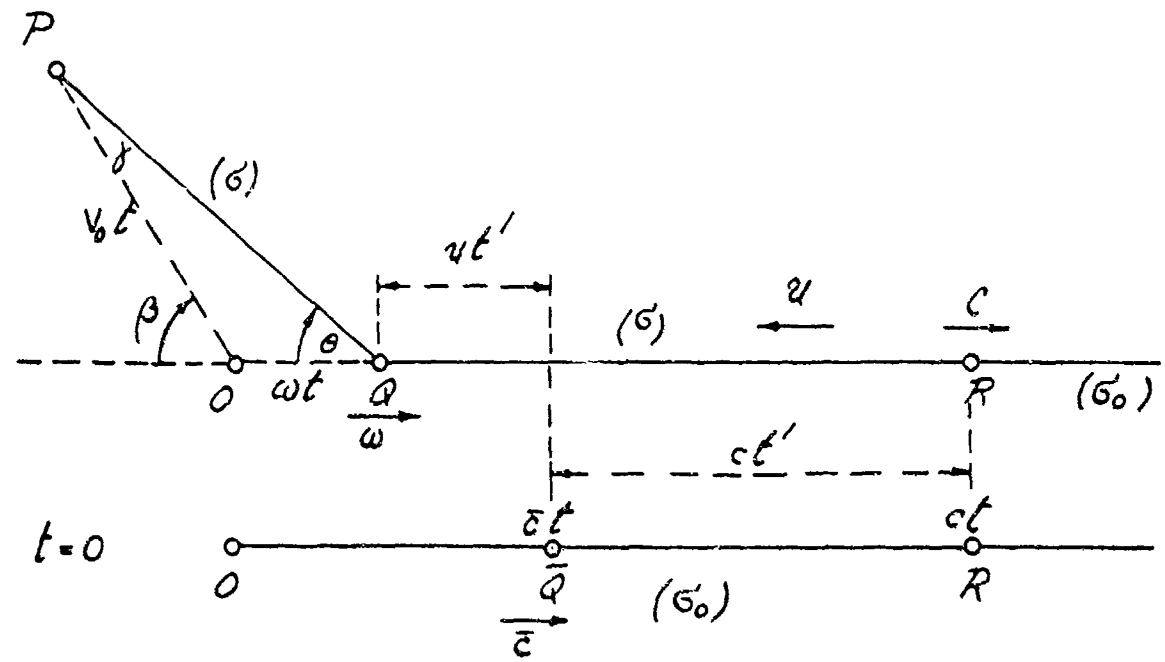


FIGURE 30

The initial cable length which is influenced by the motion at the time t is $l = OR = ct$. The elongation at this time is $\varepsilon = PQ - OQ$. From the triangle OPQ the following is obtained:

$$PQ = \sqrt{V_0^2 t^2 + \omega^2 t^2 + 2\omega V_0 t^2 \cos\beta}$$

Thus

$$\frac{\varepsilon}{l} = \frac{\varepsilon}{ct} = \sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\,\frac{V_0}{c}\,\frac{\omega}{c}\cos\beta} - \frac{\omega}{c}$$

According to Hooke's law (formula (6)) we obtain

$$\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}} = \frac{u}{c} = \sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\,\frac{V_0}{c}\,\frac{\omega}{c}\cos\beta} - \frac{\omega}{c}. \qquad (158)$$

Here $\frac{u}{c}$ and $\frac{\omega}{c}$ are functions of σ according to (156), (157) and (145). We write now formula (158) in the form

$$\left(\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}}\right)^2 + 2\,\frac{\frac{\sigma-\sigma_0}{E}}{1+\frac{\sigma_0}{E}}\,\frac{\omega}{c} = \left(\frac{V_0}{c}\right)^2 + 2\,\frac{V_0}{c}\,\frac{\omega}{c}\cos\beta$$

and replace $\frac{\omega}{c}$ by the right side of (157) which yields

$$\left(\frac{V_o}{c}\right)^2 + 2\frac{V_o}{c}\left(\frac{\bar{c}}{c} - \frac{u}{c}\left(1-\frac{\bar{c}}{c}\right)\right)\cos\beta$$

$$= \left(\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\right)^2 + 2\,\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\left(\frac{\bar{c}}{c} - \frac{u}{c}\left(1-\frac{\bar{c}}{c}\right)\right).$$

Because of (156) and (145) this yields

$$\left(\frac{V_o}{c}\right)^2 + 2\frac{V_o}{c}\left(\sqrt{\frac{\frac{\sigma}{E}}{1+\frac{\sigma}{E}}} - \frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\left(1-\sqrt{\frac{\frac{\sigma}{E}}{1+\frac{\sigma}{E}}}\right)\right)\cos\beta$$

$$= 2\,\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\left(1+\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\right)\sqrt{\frac{\frac{\sigma}{E}}{1+\frac{\sigma}{E}}} - \left(\frac{\frac{\sigma-\sigma_o}{E}}{1+\frac{\sigma_o}{E}}\right)^2. \tag{159}$$

This equation determines the oblique impact stress σ if the impact velocity V_o, the initial stress σ_o, the elasticity modulus E and the density ρ at the initial stress σ_o are given, c being determined by

$$c = \sqrt{\left(1+\frac{\sigma_o}{E}\right)\frac{E}{\rho}}$$

according to formula (141). This result can be simplified if we introduce

$$c_o = \sqrt{\frac{E}{\rho_o}}$$

where ρ_o is the density at zero stress. Then

$$\frac{c}{c_o} = \sqrt{\frac{\rho_o}{\rho}}\,\sqrt{1+\frac{\sigma_o}{E}}\,.$$

Since

$$\frac{\rho_o}{\rho} = 1+\frac{\sigma_o}{E}$$

according to formula (15) we have

$$\frac{c}{c_o} = 1+\frac{\sigma_o}{E}\,. \tag{160}$$

If we multiply now equations (159) by $\left(\frac{c}{c_0}\right)^2$ and use this relation we obtain the final result:

The impact velocity V_0, the impact angle β, and the impact stress σ are connected by the equation

$$\boxed{\left(\frac{V_0}{c_0}\right)^2 + 2\frac{V_0}{c_0}\left(\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E}\right)\cos\beta = 2\frac{\sigma-\sigma_0}{E}\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} - \left(\frac{\sigma-\sigma_0}{E}\right)^2} \tag{161}$$

where σ_0 is the initial stress, E the elasticity modulus and $c_0 = \sqrt{\frac{E}{\rho_0}}$ the longitudinal wave velocity, ρ_0 being the mass density at zero stress.

It can be expected that for $\beta = 0$ equation (161) yields the formula for the longitudinal impact In this case

$$\left(\frac{v_0}{c_0} - \frac{\sigma-\sigma_0}{E}\right)^2 + 2\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}}\left(\frac{v_0}{c_0} - \frac{\sigma-\sigma_0}{E}\right) = 0$$

Thus either

$$\frac{\sigma-\sigma_0}{E} = \frac{V_0}{c_0} \tag{162}$$

or

$$\frac{\sigma-\sigma_0}{E} - 2\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} = \frac{V_0}{c_0}.$$

The first of these equations is identical with the formula for the longitudinal impact. The second is meaningless because it does not have a real solution σ for a positive value of V_0.

The formula for the transverse (perpendicular) impact is obtained for $\beta = 90°$ which yields

$$\left(\frac{V_0}{c_0}\right)^2 = 2\frac{\sigma-\sigma_0}{E}\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} - \left(\frac{\sigma-\sigma_0}{E}\right)^2. \tag{163}$$

If β increases toward the limit 180° the formula for the reverse impact follows from (161). We have now

$$\left(\frac{V_0}{c_0} + \frac{\sigma-\sigma_0}{E}\right)^2 = 2\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}}\left(\frac{V_0}{c_0} + \frac{\sigma-\sigma_0}{E}\right).$$

Thus either

$$\frac{\sigma-\sigma_0}{E} = -\frac{v_0}{c_0}$$

or

$$2\sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E} = \frac{v_0}{c_0}\,. \tag{164}$$

The first of these equations corresponds to the longitudinal impact with negative velocity (compression) which is meaningless for a cable while formula (164) represents the actual formula for the reverse impact.

We complete these relations by listing for the kink velocity in the general case the formula

$$\frac{\omega}{c_0} = \sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E} \tag{165}$$

which follows from (157) by multiplication with $\frac{a}{c_0} = 1 + \frac{\sigma_0}{E}$ and for the kink angle Θ the formulas

$$\operatorname{ctg}\Theta = \frac{\frac{\omega}{c_0} + \cos\beta}{\sin\beta}\,, \qquad \sin\Theta = \frac{V_0}{a}\sin\beta \tag{166}$$

which follow from the triangle OPQ in Figure 30.

In the case of the reverse impact (164) and (165) yield

$$\frac{V_0}{c_0} - \frac{\omega}{c_0} = \sqrt{\frac{\sigma}{E}}\sqrt{1+\frac{\sigma}{E}} = \frac{1}{2}\left(\frac{u}{c_0} + \frac{V_0}{c_0}\right)$$

Thus

$$\frac{\omega}{c_0} = \frac{1}{2}\left(\frac{V_0}{c_0} - \frac{u}{c_0}\right).$$

This result shows that the 180° kink moves with a velocity ω which is smaller than $\frac{V_0}{2}$. This must be so because mass is picked up continuously during the motion resulting in stress which runs ahead of the kink along the cable. Only in the case of a massless cable the kink would move with the velocity $\frac{V_0}{2}$.

For other impact problems compare references 21 and 22.

6. Approximations and Evaluations

If $\frac{\sigma}{E}$ and $\frac{\sigma_0}{E}$ are negligibly small compared with unity formula (161) can be simplified to

$$\left(\frac{V_0}{c_0}\right)^2 + 2\frac{V_0}{c_0}\left(\sqrt{\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E}\right)\cos\beta = 2\frac{\sigma-\sigma_0}{E}\sqrt{\frac{\sigma}{E}} - \left(\frac{\sigma-\sigma_0}{E}\right)^2. \tag{166}$$

This and the following formulas up to (171) are sufficiently correct for conventional steel cables and steel cables with hemp core. (166) yields

$$\frac{V_0}{c_0} = -\left(\sqrt{\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E}\right)\cos\beta + \sqrt{\frac{\sigma}{E} - \left(\sqrt{\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E}\right)^2\sin^2\beta}. \tag{167}$$

The corresponding approximations for the longitudinal and transverse wave velocities are

$$c = c_0 = \sqrt{\frac{E}{\rho_0}}, \qquad \bar{c} = \sqrt{\frac{\sigma}{\rho_0}}, \qquad \rho = \rho_0. \tag{168}$$

The particle velocity is determined by

$$\frac{u}{c_0} = \frac{\sigma-\sigma_0}{E} \tag{169}$$

and the kink velocity by

$$\frac{\omega}{c_0} = \sqrt{\frac{\sigma}{E}} - \frac{\sigma-\sigma_0}{E} \tag{170}$$

The kink angle Θ is given correctly as before by

$$\operatorname{ctg}\Theta = \frac{\frac{\omega}{V_0} + \cos\beta}{\sin\beta}, \qquad \sin\Theta = \frac{V_0}{\bar{c}}\sin\beta. \tag{171}$$

Equation (167) can be evaluated for all data by one simple graph. For this purpose all lengths of the upper Figure 30 have to be measured in the unit $c_0 t$ so that $OR = 1$. Then $OP = \frac{V_0}{c_0}$ and $OQ = \frac{\omega}{c_0}$ (see Figure 31).

Let the circle with the center Q and the radius $\frac{\omega}{c_0}$ cut the side PQ of the triangle OPQ in S. Then PS is the elongation of the cable divided by $c_0 t$. Thus

$$PS = \frac{\sigma-\sigma_0}{E}.$$

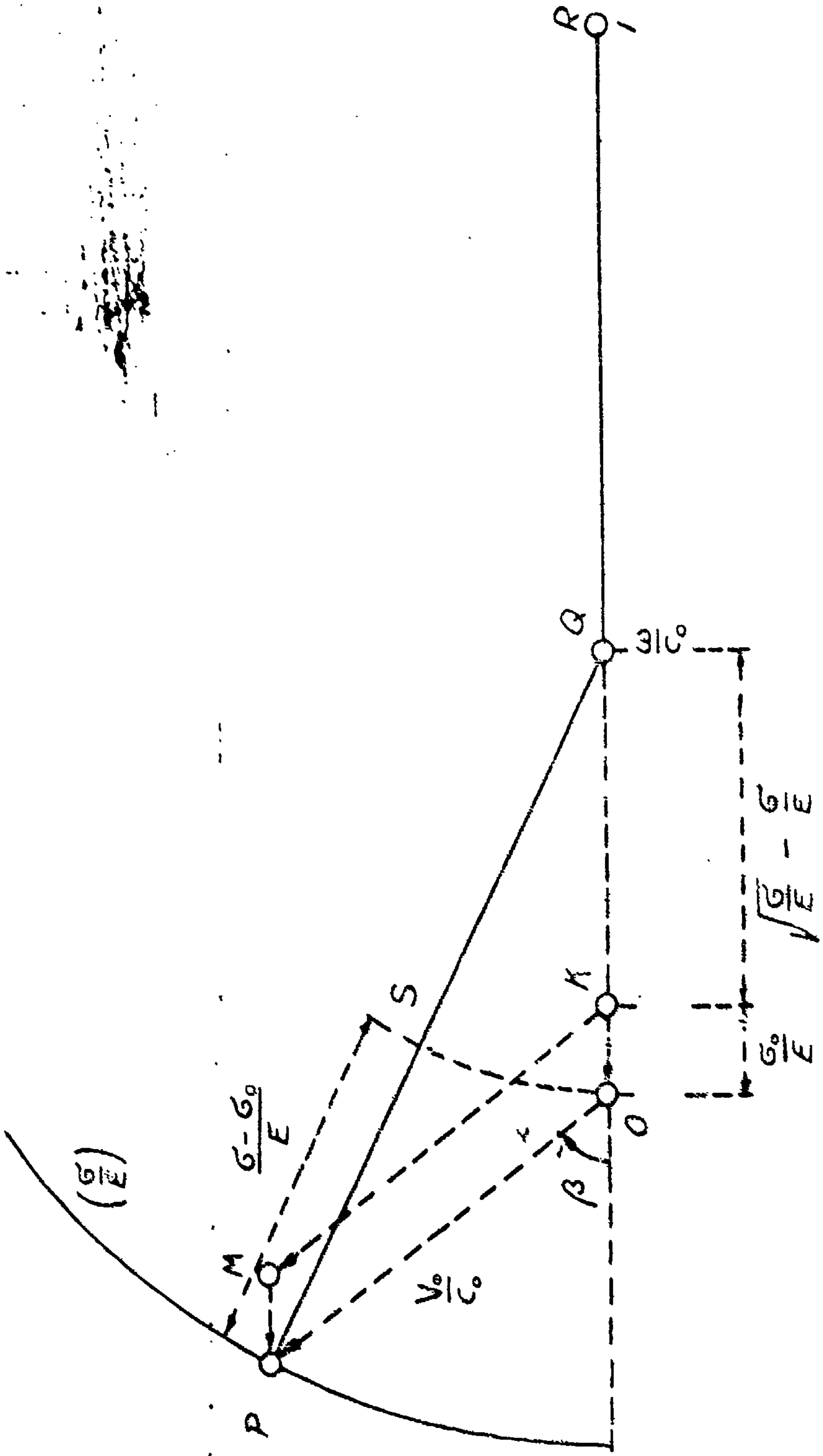
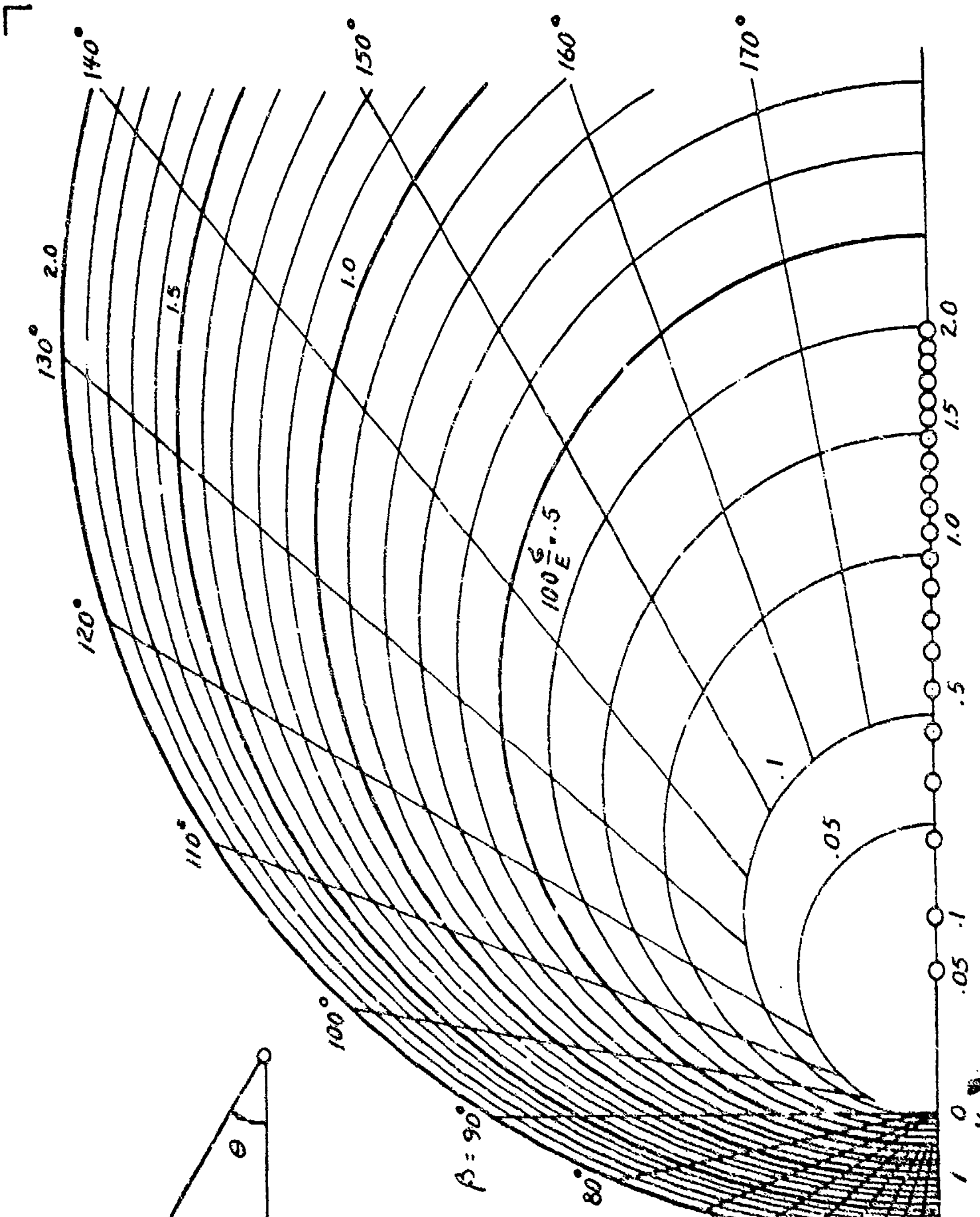


FIGURE 31



OBLIQUE IMPACT STRESS

FIGURE 32

FIGURE 32a
OBLIQUE IMPACT STRESS (DETAIL)

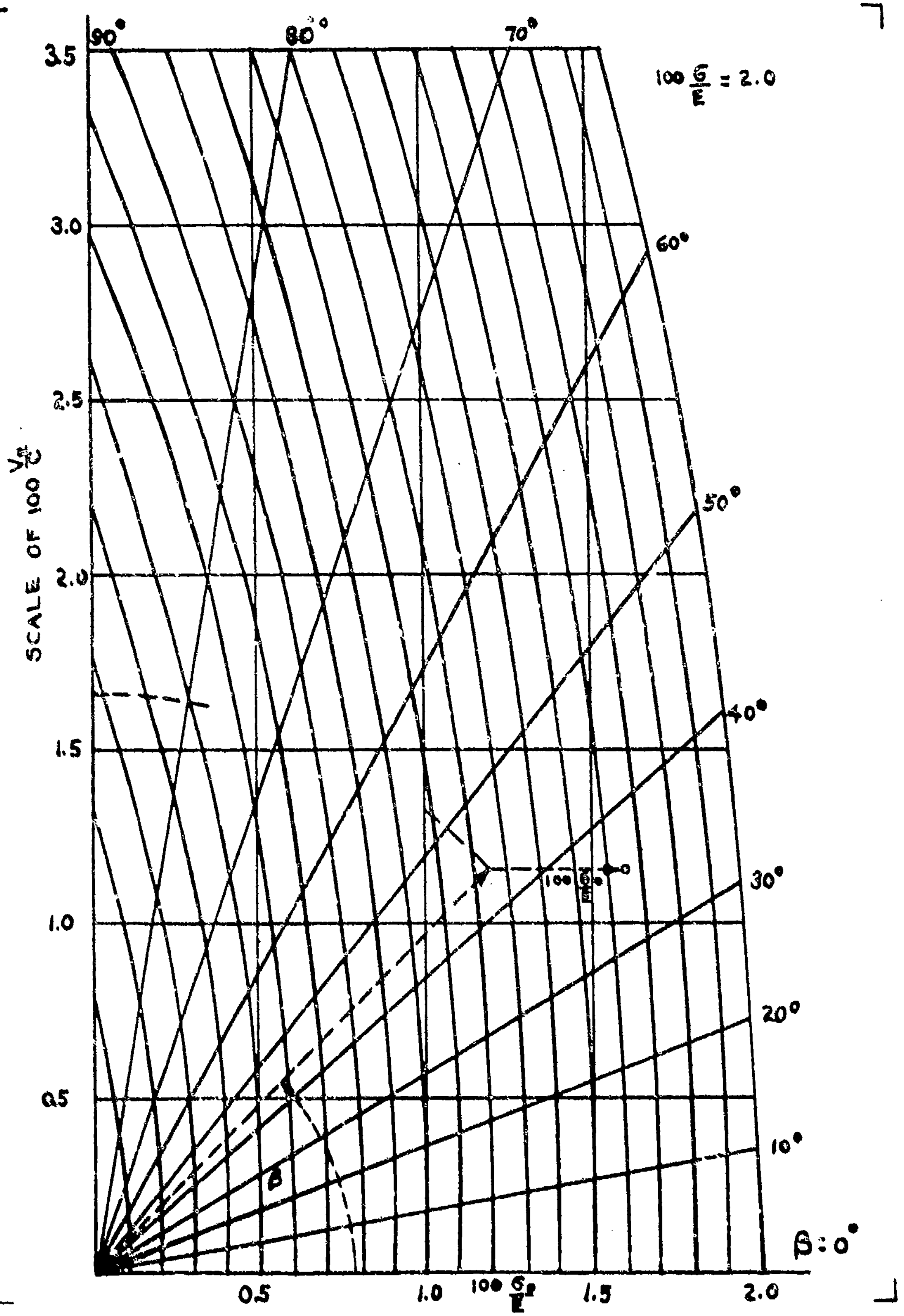
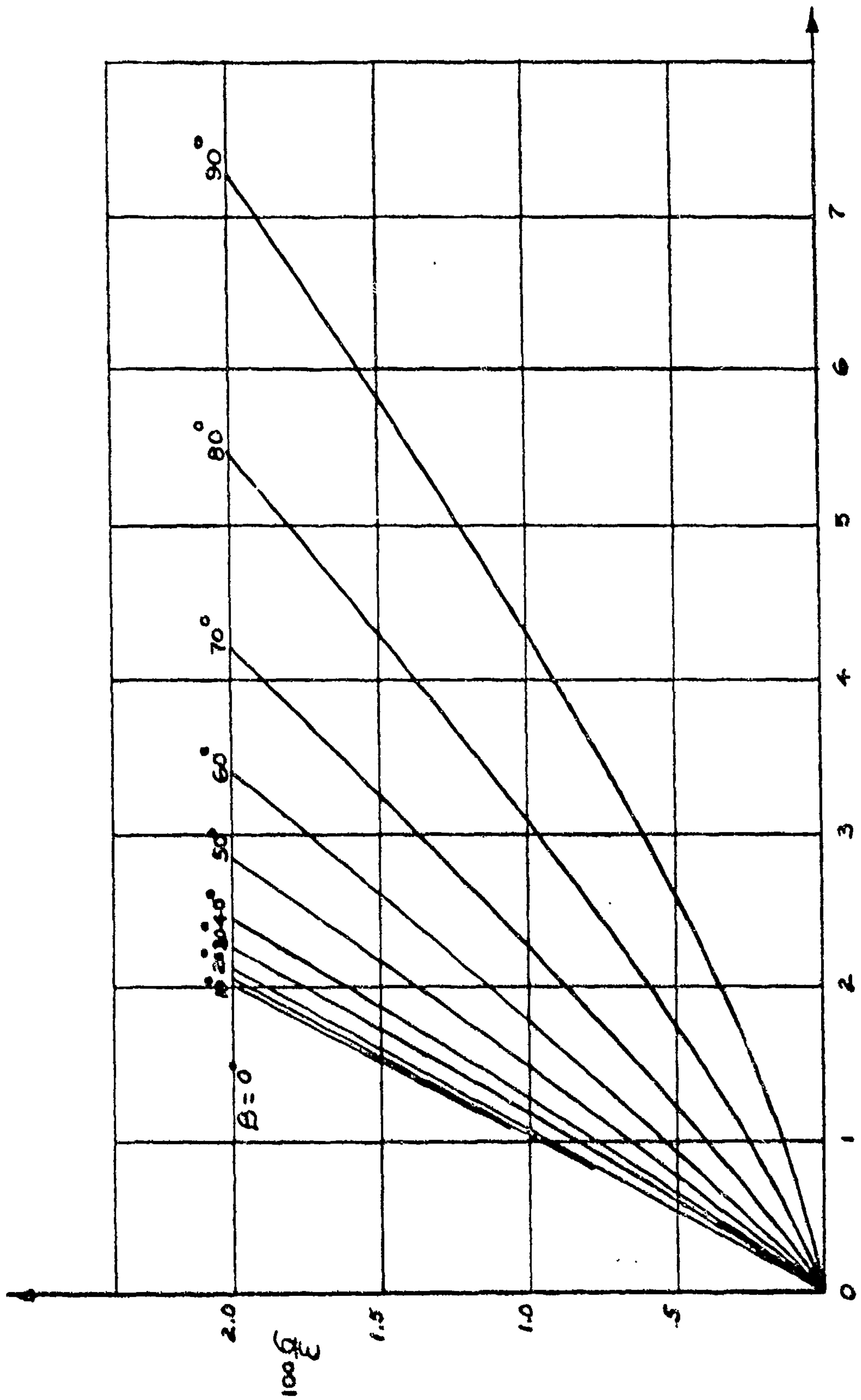


FIGURE 32b
OBLIQUE IMPACT STRESS FOR ZERO INITIAL STRESS



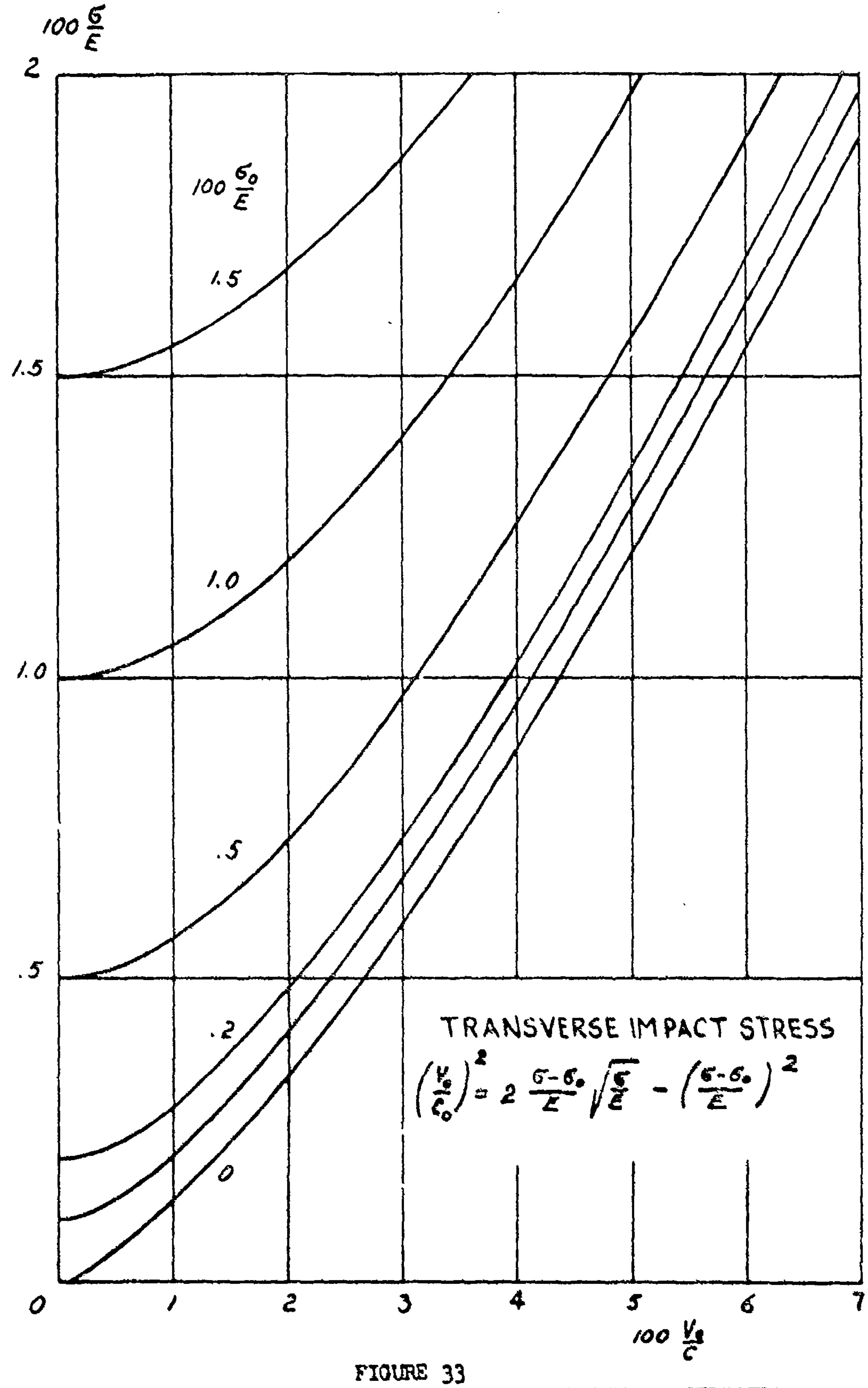


FIGURE 33
TRANSVERSE IMPACT STRESS FOR VARIOUS INITIAL STRESSES

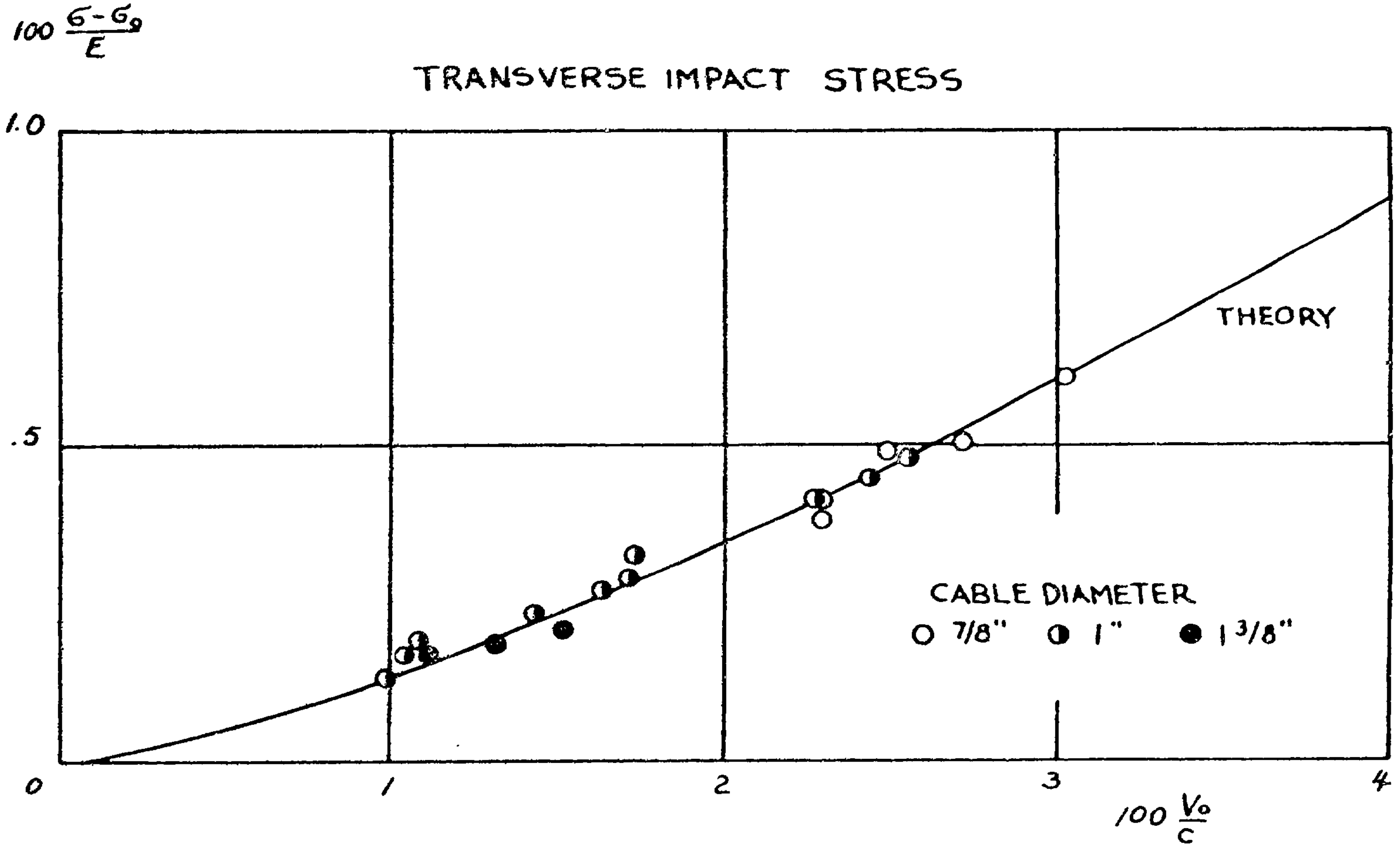
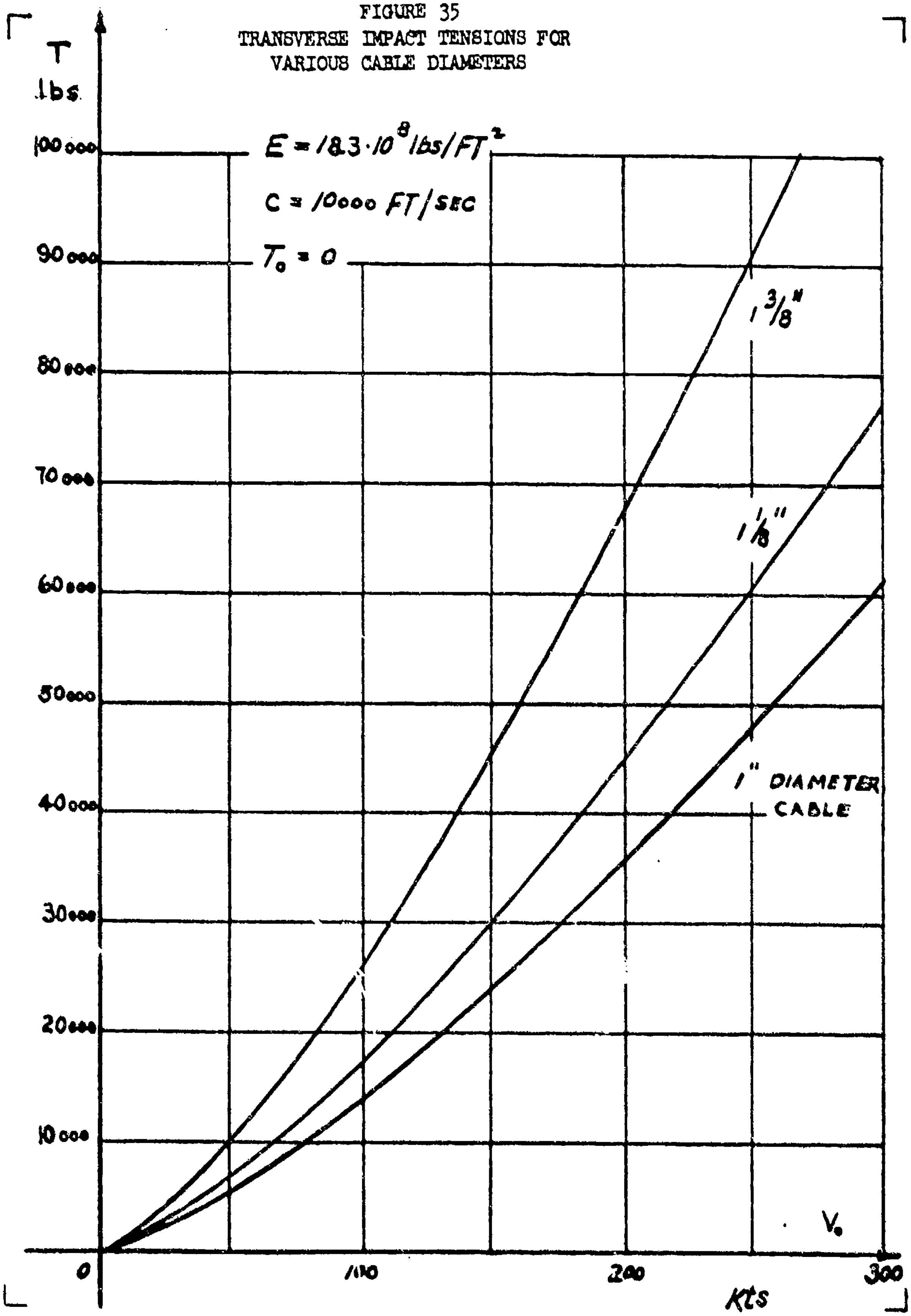


FIGURE 34
MEASUREMENTS OF THE TRANSVERSE IMPACT STRESS

FIGURE 35
TRANSVERSE IMPACT TENSIONS FOR
VARIOUS CABLE DIAMETERS



Now
$$PQ = \sqrt{\frac{\sigma}{E}}\,.$$

We choose K in the distance $\frac{\sigma_0}{E}$ from O as a fixed point for the following construction: Computing for various values of $\frac{\sigma}{E}$ the expression $\sqrt{\frac{\sigma}{E}} - \frac{\sigma}{E}$ we construct the circles with the radii $\sqrt{\frac{\sigma}{E}}$ and the centers Q in the distance $\sqrt{\frac{\sigma}{E}} - \frac{\sigma}{E}$ from K and denote these circles by their corresponding value $\left(\frac{\sigma}{E}\right)$. We construct from K the vector with the length $\frac{V_0}{c_0}$ under the angle β against KO up to point M and from M the vector $\frac{\sigma_0}{E}$ parallel to KO. Then the endpoint of this vector is the point P situated on the circle $\left(\frac{\sigma}{E}\right)$. A sufficiently dense set of circles yields a graph from which the oblique impact tensions can be determined for any given data. Figure 32 shows this graph. Figure 32a shows a detail of Figure 32. In Figure 32b, the curves $\beta =$ constant are plotted for zero initial stress.

In the case of perpendicular impact $(\beta = 90^\circ)$ formula (169) yields
$$\left(\frac{V_0}{c_0}\right)^2 = 2\,\frac{\sigma - \sigma_0}{E}\sqrt{\frac{\sigma}{E}} - \left(\frac{\sigma - \sigma_0}{E}\right)^2. \tag{172}$$

Figure 33 represents this relation between $\frac{\sigma}{E}$ and $\frac{V_0}{c_0}$ for various values of $\frac{\sigma_0}{E}$ in order to show the influence of the initial stress on the resulting impact stress.

If $\frac{\sigma - \sigma_0}{E}$ is negligibly small compared with $\sqrt{\frac{\sigma}{E}}$ then equation (166) yields a further approximation
$$\left(\frac{V_0}{c_0}\right)^2 + 2\,\frac{V_0}{c_0}\sqrt{\frac{\sigma}{E}}\,\cos\beta = 2\,\frac{\sigma - \sigma_0}{E}\sqrt{\frac{\sigma}{E}}\,. \tag{173}$$
In the special case of perpendicular impact $(\beta = 90^\circ)$ and a σ_0 which is zero or negligibly small compared with σ this equation yields
$$\boxed{\frac{\sigma - \sigma_0}{E} = \left(\frac{1}{2}\right)^{\frac{2}{3}}\left(\frac{v_0}{c_0}\right)^{\frac{4}{3}}.} \tag{174}$$

This formula was first derived by the author of the present monography in 1948 (reference (8)). The curve, Figure 34, represents this relation and shows measured values* obtained from different cables, initial tensions and impact velocities (compare also references (13) and (16)). Figure 35 shows the transverse impact tensions in pounds as functions of the impact velocity in knots for frequently used cable data.

* Mainly NAEF measurements. For the technique of impact measurements compare reference 20.

7. Energy Relations

As in the case of longitudinal impact, it is of interest to study the distribution of the input energy on the moving cable parts in the general case of oblique impact and especially in the case of perpendicular impact.

There have to be considered now three parts into which the work done at the cable end point O will be split, the kinetic energy of the cable segment PQ (see Figure 30), the kinetic energy of the segment QR and the potential energy stored in both segments in form of stress. The mass m_1 of PQ at the time t is the same as the mass of $O\bar{Q}$ at the time zero which is $m_1 = \rho q \bar{c} t$. The segment PQ moves with the velocity V_0. Thus its kinetic energy is

$$H_1 = \rho q \bar{c} t \cdot \frac{V_0^2}{2}. \tag{175}$$

The cable segment QR has at the time t the same mass as $\bar{Q}R$ at the time zero which is $m_2 = \rho q (c - \bar{c}) t$. Its velocity is the particle velocity u. Therefore, its kinetic energy

$$H_2 = \rho q (c - \bar{c}) t \cdot \frac{u^2}{2}. \tag{176}$$

The work required to elongate the cable statically from the stress σ_0 to the stress σ is

$$H_\sigma = \frac{\sigma + \sigma_0}{2} q \cdot \varepsilon$$

where ε is the elongation. Now $\frac{\varepsilon}{ct} = \frac{u}{c}$. Thus (177)

$$H_\sigma = \frac{\sigma + \sigma_0}{2} q c t \cdot \frac{u}{c}.$$

We use dimensionless energy coefficients dividing the energies by the arbitrarily chosen energy

$$H_0 = \rho q c t \cdot \frac{c^2}{2} \tag{178}$$

and obtain

$$\eta_1 = \frac{H_1}{H_0} = \frac{\bar{c}}{c}\left(\frac{V_0}{c}\right)^2, \tag{179}$$

$$\eta_2 = \frac{H_2}{H_0} = \left(1-\frac{\bar{c}}{c}\right)\left(\frac{u}{c}\right)^2, \tag{180}$$

$$\eta_\sigma = \frac{H_\sigma}{H_0} = \frac{\sigma+\sigma_0}{E} \; \frac{1}{1+\frac{\sigma_0}{E}} \; \frac{u}{c}. \tag{181}$$

Using the formulas (145) and (156) we can express $\frac{\sigma}{E}$ and $\frac{\sigma_0}{E}$ by $\frac{u}{c}$ and $\frac{\bar{c}}{c}$. We get

$$\frac{\sigma}{E} = \frac{\left(\frac{\bar{c}}{c}\right)^2}{1-\left(\frac{\bar{c}}{c}\right)^2}, \qquad \frac{\sigma_0}{E} = \frac{\frac{\left(\frac{\bar{c}}{c}\right)^2}{1-\left(\frac{\bar{c}}{c}\right)^2} - \frac{u}{c}}{1+\frac{u}{c}}.$$

Then η_σ takes the form

$$\eta_\sigma = 2\left(\frac{\bar{c}}{c}\right)^2 \frac{u}{c}\left(1+\frac{u}{c}\right) - \left(\frac{u}{c}\right)^2 \tag{182}$$

a form which corresponds in the variables to η_1 and η_2.

For the total energy $\eta = \eta_1 + \eta_2 + \eta_\sigma$ we obtain now

$$\eta = \frac{\bar{c}}{c}\left(\left(\frac{v_0}{c}\right)^2 - \left(\frac{u}{c}\right)^2\right) + 2\left(\frac{\bar{c}}{c}\right)^2 \frac{u}{c}\left(1+\frac{u}{c}\right). \tag{183}$$

We must find the same value η by computation of the work done against the tension by moving the cable end point O in the direction β with the velocity V_0. This work is equal (compare Figure 30)

$$H = q\,\sigma\cos\gamma \cdot v_0 t \tag{184}$$

or in dimensionless form

$$\eta = 2\frac{\sigma}{E} \; \frac{1}{1+\frac{\sigma_0}{E}} \; \frac{V_0}{c}\cos\gamma. \tag{185}$$

From

$$\cos\gamma = \cos(\beta-\theta) = \cos\beta\cos\theta + \sin\beta\sin\theta$$

and formula (171)

$$\operatorname{ctg}\theta = \frac{\frac{\omega}{V_0} + \cos\beta}{\sin\beta}$$

follows

$$\cos\gamma = \frac{\frac{V_0}{c} + \frac{\omega}{c}\cos\beta}{\sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\frac{V_0}{c}\frac{\omega}{c}\cos\beta}}. \tag{186}$$

Formulas (158) and (157) yield

$$\sqrt{\left(\frac{V_0}{c}\right)^2+\left(\frac{\omega}{c}\right)^2+2\,\frac{V_0}{c}\,\frac{\omega}{c}\cos\beta} = \frac{\omega}{c}+\frac{u}{c} = \frac{\bar{c}}{c}\left(1+\frac{u}{c}\right)$$

and

$$2\,\frac{V_0}{c}\,\frac{\omega}{c}\cos\beta = 2\,\frac{\bar{c}}{c}\,\frac{u}{c}\left(1+\frac{u}{c}\right)-\left(\frac{V_0}{c}\right)^2-\left(\frac{u}{c}\right)^2.$$

Therefore, because of (185),

$$\eta = \frac{\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}\;\frac{\left(\frac{V_0}{c}\right)^2-\left(\frac{u}{c}\right)^2+2\,\frac{\bar{c}}{c}\,\frac{u}{c}\left(1+\frac{u}{c}\right)}{\frac{\bar{c}}{c}\left(1+\frac{u}{c}\right)}. \tag{187}$$

Formula (145) for $\frac{\bar{c}}{c}$ shows that

$$\frac{\frac{\sigma}{E}}{\frac{\bar{c}}{c}} = \frac{\bar{c}}{c}\left(1+\frac{\sigma}{E}\right)$$

and (156) for $\frac{u}{c}$ that

$$1+\frac{u}{c} = \frac{1+\frac{\sigma}{E}}{1+\frac{\sigma_0}{E}}.$$

Therefore the expression (187) for η becomes

$$\eta = \frac{\bar{c}}{c}\left(\left(\frac{V_0}{c}\right)^2-\left(\frac{u}{c}\right)^2\right)+2\left(\frac{\bar{c}}{c}\right)^2\frac{u}{c}\left(1+\frac{u}{c}\right) \tag{188}$$

which is identical with (183).

Formulas (179), (180) and (182) show correctly the distribution of the input energy over the cable in the general case as functions of the stress σ if $\frac{V_0}{c}$, $\frac{\bar{c}}{c}$ and $\frac{u}{c}$ are replaced by their expressions in σ.

If now $\frac{\sigma}{E}$ and $\frac{\sigma_0}{E}$ are negligibly small compared with 1 these formulas simplify to

$$\begin{aligned}\eta_1 &= \sqrt{\frac{\sigma}{E}}\left(\frac{V_0}{c}\right)^2,\\ \eta_2 &= \left(1-\sqrt{\frac{\sigma}{E}}\right)\left(\frac{\sigma-\sigma_0}{E}\right)^2,\\ \eta_\sigma &= 2\,\frac{\sigma}{E}\,\frac{\sigma-\sigma_0}{E}-\left(\frac{\sigma-\sigma_0}{E}\right)^2.\end{aligned} \tag{189}$$

Thus

$$\frac{\eta_i}{\eta_\sigma} = \left(1 - \sqrt{\frac{\sigma}{E}}\right) \frac{\sigma - \sigma_0}{\sigma + \sigma_0} \tag{190}$$

If, moreover, σ_0 is zero or small compared with σ the last formula shows the strain energy is always approximately equal to the kinetic energy of the longitudinally moving cable segment.

In the case of perpendicular impact we have because of (173) the following approximate expression:

$$\left(\frac{v_0}{c}\right)^2 = 2\,\frac{\sigma - \sigma_0}{E}\sqrt{\frac{\sigma}{E}}$$

and therefore, approximately

$$\eta_1 = 2\left(\frac{\sigma - \sigma_0}{E}\right)^2$$

while approximately,
$$\eta_\sigma = \left(\frac{\sigma - \sigma_0}{E}\right)^2.$$

So the kinetic energy of the transverse moving segment is about 50 percent of the total energy while the kinetic energy of the longitudinally moving segment and the strain energy are about 25 percent each of the total energy. The last result has been found by J. Thomlinson (England). Formula (190), however, shows that for larger stresses those statements can be considered as crude approximations only because $\sqrt{\frac{\sigma}{E}}$ is not always negligibly small compared with 1 if $\frac{\sigma}{E}$ is so.

8. Impact at a Moving Cable

The purpose of the investigations of this and the following sections is to show that the oblique impact formula is not restricted in its applications to the special problem of determining the stress in a cable due to impact but that it is a convenient tool for the solution of many more general problems. This becomes obvious if we realize that any cable motion can be replaced, with any desired degree of accuracy, by the motion of an equivalent polygon-shaped cable whereon each side, in a given moment, the velocity and the stress have constant values. The resulting configuration and stress distribution may be computed then for any subsequent time as the result of a sequence of impacts. If the number of steps required for the solution of a particular problem in this way is not too large, it can be solved by a sequence of readings from the graph for the oblique impact formula. This method will be illustrated in this chapter at a series of problems, frequently occurring with the practical use of cables.

First we consider a straight cable SR with the constant stress σ_1, which moves longitudinally with the constant velocity u_1, (see Figure 36). We assume that a point Q of the moving cable is subject to an impact with the velocity V_0 under the angle β relative to the moving cable.

The problem of determining the stress σ resulting from this impact can be computed in two ways:

a. We can assume that we are moving with the cable longitudinally with the velocity u_1 (see Figure 35a). Then the impact at the cable segment QR is simply equivalent to the oblique impact with the velocity V_0 under the angle β at the cable in rest with the initial stress σ_1 and the impact stress σ follows directly from the impact formula or the graph.

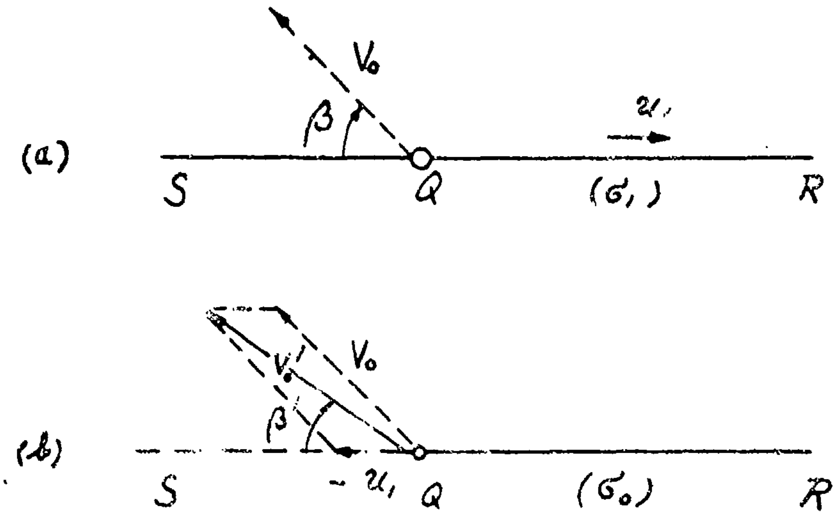


FIGURE 36

b. We can assume that the longitudinal motion has been produced by a longitudinal impact with the velocity u_1 at a cable in rest with a suitable initial stress σ_0 so that

$$\frac{\sigma_1 - \sigma_0}{E} = \frac{u_1}{c_0}$$

according to the longitudinal impact formula, which determines the unknown σ_0. The stress σ produced by the impact velocity V_0 in the direction β will then be the same as the stress produced by the impact velocity V_0' in the direction β' at the cable in rest with the initial stress σ_0 where V_0' and β' are shown in Figure 36b, V_0' being the resultant of the velocities V_0 and $-u_1$. The identity of both results follows directly from the graph Figure 32.

The impact stress in the cable segment QS can be determined in the same way replacing β by $180° - \beta$. The stresses in the two segments will be different in general because no stress is allowed to pass over the moving point Q in this case.

This example shows how the determination of the impact stress at a moving cable can be reduced to the determination of the impact stress at a cable in rest. In a similar way the problem of determination of the impact stress can be solved if the cable initially performs any motion parallel to itself with a constant velocity.

9. Sequence of Impacts

The problem to be solved in this section is the following. The endpoint O of a cable in rest with the initial stress σ_o is assumed to move during a time t_o with a constant velocity V_o under the angle β_o into the position P_o (see Figure 37). After that time point P_o is assumed to move with a constant velocity V_1 under an angle γ_1 against the original position of the cable during a time t_1 into the position P_1. Due to the first of these impacts the cable will obtain a kink which will be situated at the time t_o at a point Q_o as indicated in Figure 37. The second impact will produce a kink on the cable segment $P_o Q_o$ which will be situated at the time $t_o + t_1$ at a point Q_1. To be determined are the velocity $\bar{V}_1$ and the angle β_1 relative to the moving cable segment $P_o Q_o$ which produce the second impact guiding P_o to P_1.

The first impact with the velocity V_o in the direction β_o produces in the cable with the initial stress σ_o a stress σ_1 which can be determined from the oblique impact formula or the graph. The corresponding kink angle θ_o is known from one of the formulas (166) and can be read from the graph too. The second impact has, with respect to the moving segment $P_o Q_o$ the unknown velocity $\bar{V}_1$ and the direction β_1. Without this impact P_o

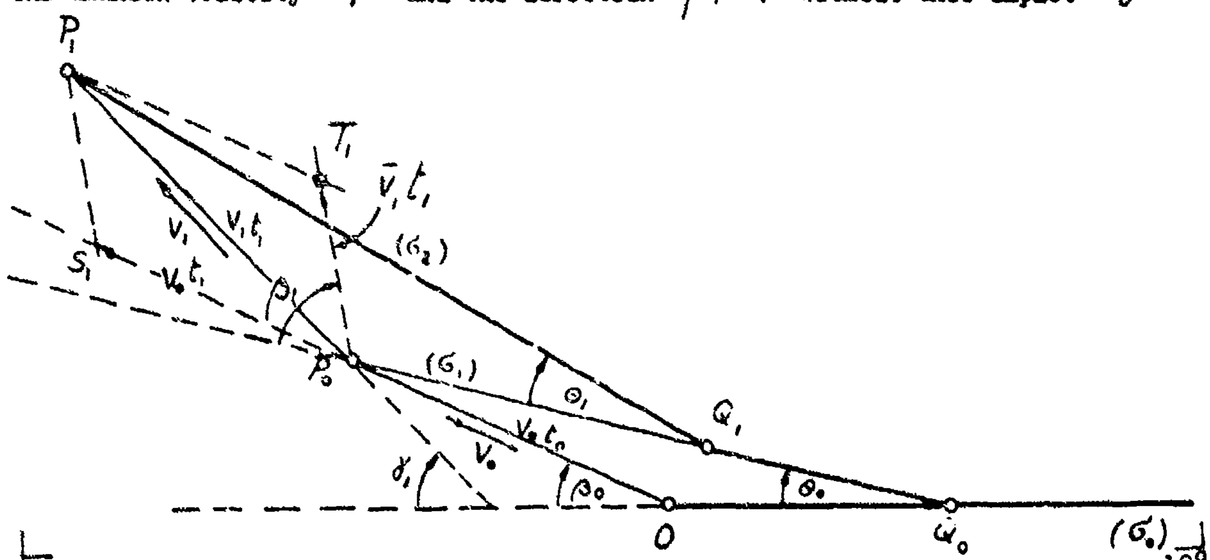


FIGURE 37

would move during the time t_1 into the position S_1. The unknown impact has to force it into the position P_1. Thus the vector $\overrightarrow{P_0 T_1}$ with the length $\bar{v}_1 t_1$ under the angle β_1 and the vector $\overrightarrow{P_0 S_1}$ must have the vector $\overrightarrow{P_0 P_1}$ as resultant. Because the vectors $\overrightarrow{P_0 S_1}$ and $\overrightarrow{P_0 P_1}$ are given, the vector $\overrightarrow{P_0 T_1}$ can be constructed, $P_0 S_1 P_1 T_1$ being a parallelogram. Thus $\bar{v}_1$ and β_1 are determined.

The initial stress for the second impact is σ_1, the impact stress due to the first impact. Thus the stress σ_2 due to the second impact can be determined from the impact formula or the graph. The kink angle θ_2 follows again from one of the formula (166) or the graph.

The stress σ_2 determined in this way is valid only in a neighborhood of P_1 as long as no reflected stress wave arrives at this point. Actually the stress induced by the second impact propagates toward Q_0 and, upon arrival at this point, experiences a disturbance which will propagate forward and backward along the cable. The returning disturbance will experience another disturbance when passing over Q_1 and will finally contribute to the stress at P_1. A detailed study of the stress propagation over a kink is contained in section 11.

10. Transverse Wave Reflection at a Fixed Point

We consider an oblique impact as discussed in section 5. The initial stress of the cable in rest is assumed to be σ_0. The impact velocity is V_0 and the impact angle β (see Figure 38).

The cable segment RQ is moving longitudinally with the particle velocity u_1 determined by $\frac{\sigma_1 - \sigma_0}{E} = \frac{u_1}{c_0}$.

We consider now R as a particular cable point. It is moving with the velocity u_1 toward the kink point Q while this point moves with the kink velocity ω toward the left. If we assume that we are moving with the point R this point now is a fixed point and the kink approaches this point with the velocity $\omega + u_1$.

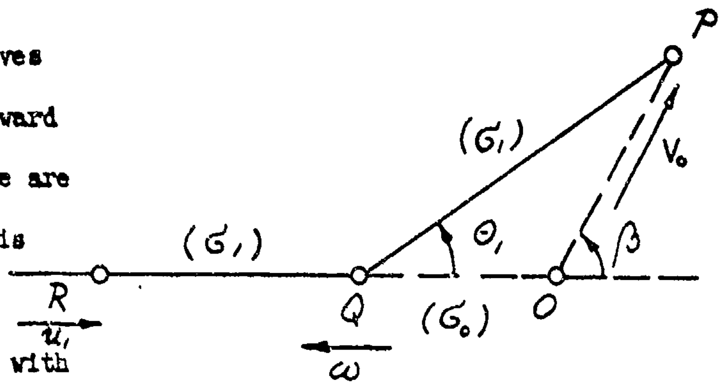


FIGURE 38

If now R would be free to move in the moment where Q coincides with R it would move in the direction RR' where R' is determined by the velocities V_0 and u_1 as shown in Figure 39. Therefore, in order to keep R in its fixed position an oblique impact with the velocity

$$v' = RR'' = RR'$$

in opposite direction to RR' has to be applied in R at the cable RP, the impact angle being β' as shown in Figure 39.

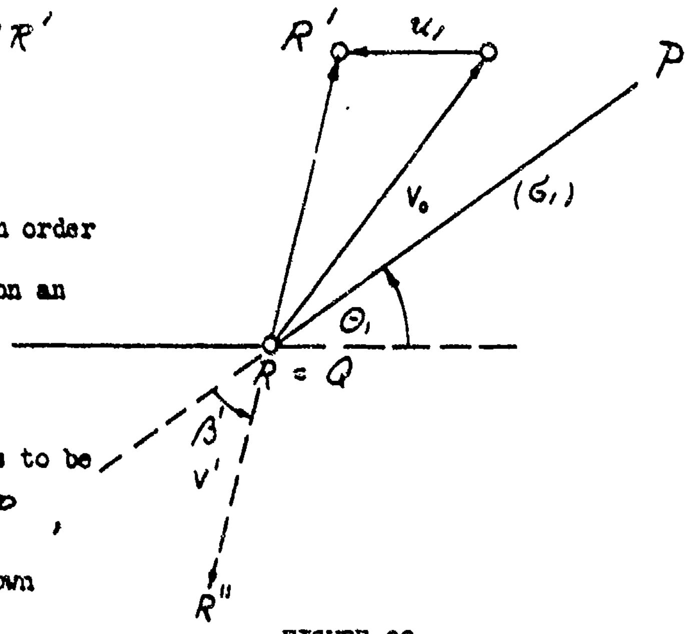
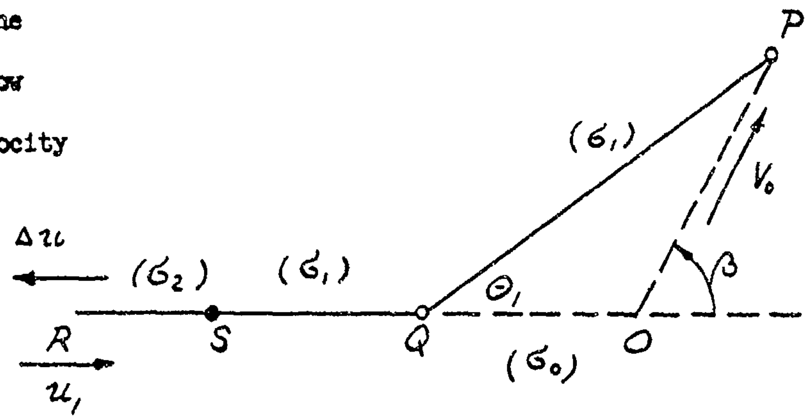


FIGURE 39

The initial stress for this impact is σ_1 . The stress σ_2 produced by this impact can be obtained from the impact formula (161) or the graph and is the stress produced by the reflection of the kink wave at the fixed point R .

11. Stress Propagation over a Kink

We consider again an oblique impact as in the preceding section (see Figure 40) using the same denotations. We now suddenly change the velocity u_1 of point R about an amount Δu producing in this way a stress σ_2 which is determined by the longitudinal impact formula



$$\frac{\sigma_2 - \sigma_1}{E} = \frac{\Delta u}{c_0}$$

FIGURE 40

The stress difference $\sigma_2 - \sigma_1$ propagates toward Q and when arriving at Q suddenly changes the motion of Q compared with that which would take place without the action of this stress difference. The disturbance of Q means an impact as well at the cable segment OP as at the segment QR. The configuration of the cable after these impacts will be as shown in Figure 41.

S' is the point which separates the stress areas σ_2 and σ_1 on RQ at a short time after the longitudinal impact Δu (see Figure 40). This point moves with the longitudinal wave velocity toward Q. From the moment on where S' reach point Q (see Figure 41),

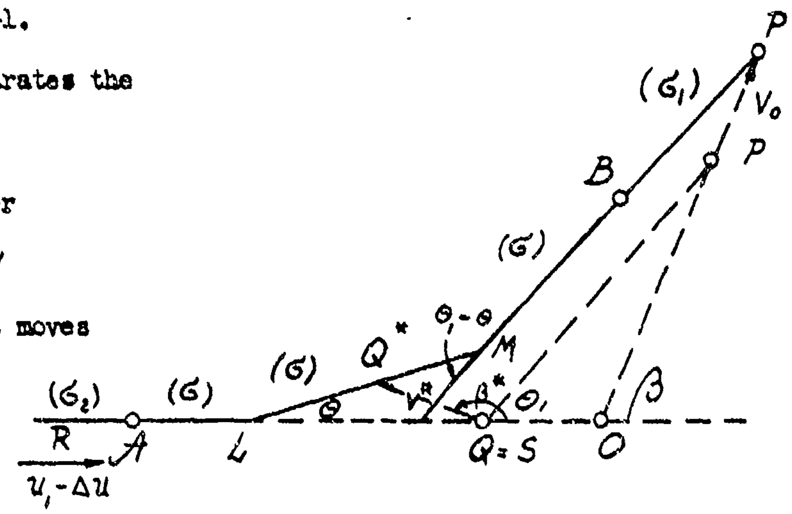


FIGURE 41

this point moves in a direction $\overrightarrow{QQ^*}$ which is unknown at the present. Two kinks L and M are forming and moving respectively toward R and P' due to the two respective impacts. Both impacts must produce the same stress according to the results of section IV 4. The stress σ propagates toward R and P' and has reached after unit time say the points A and B. Beyond A the stress is equal σ_2 and beyond B equal σ_1, assuming that the points R and P' have not been reached yet. If the kink angle at L is equal θ that at M is equal $\theta_1 - \theta$.

We denote with v' and β' the impact velocity and the impact angle with respect to the moving cable segment BP' and with v'', β'' the impact velocity and the impact angle with respect to the moving cable segment RA. Both impacts must result in the same velocity v^* and the same direction β^* for the motion of Q into the position Q^*. Both impacts must result further in the same stress σ_1 and the same angle θ. The first of these conditions yields the geometrical relation expressed by Figure 42. So if v' and β' are chosen arbitrarily v'' and β'' are determined by this configuration.

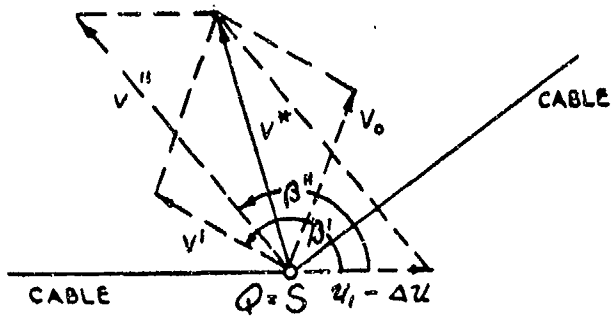
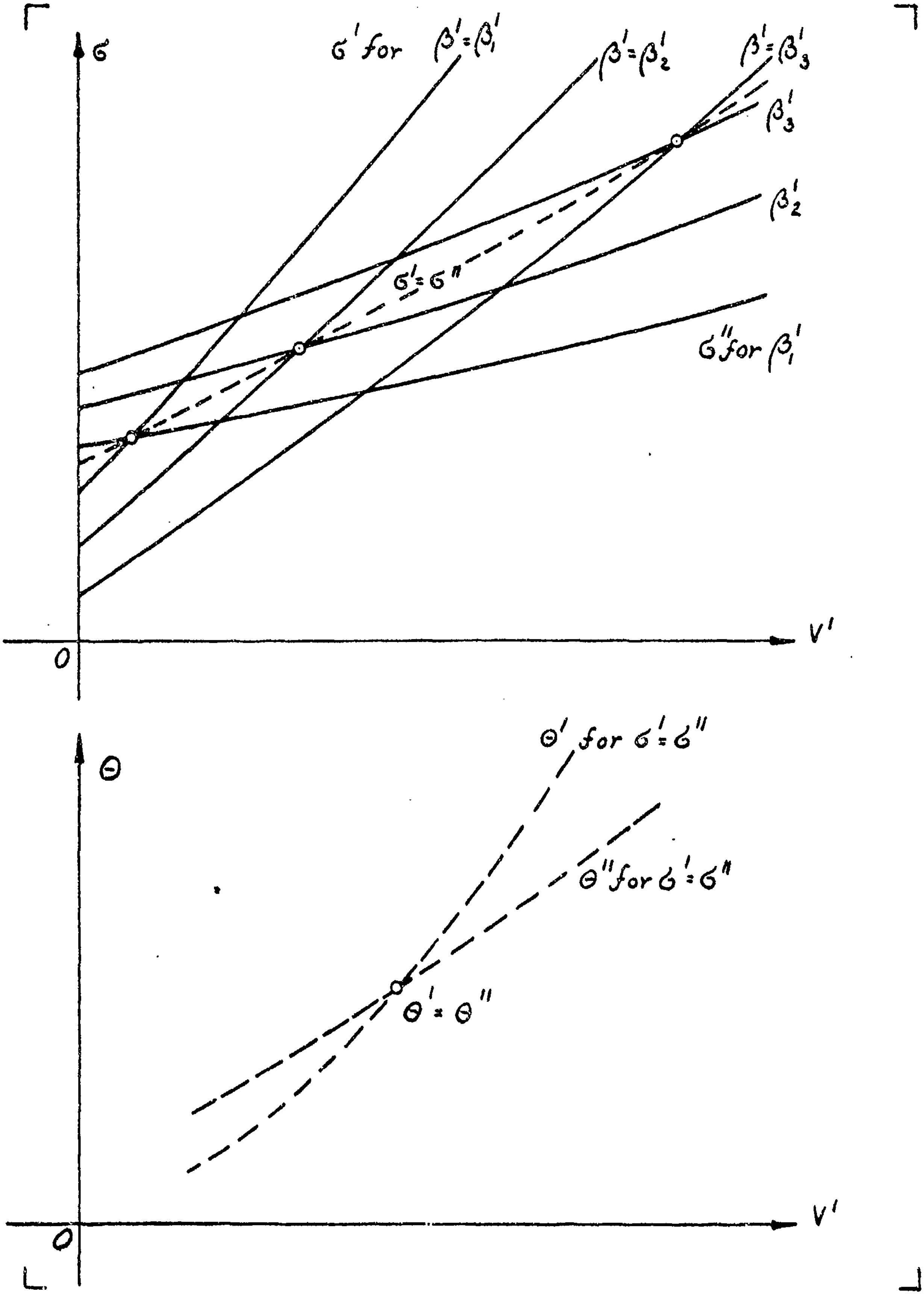


FIGURE 42

In order to satisfy the second condition, the following graphical procedure can be applied. For a fixed arbitrarily chosen value $\beta' = \beta_1'$ and a set of arbitrarily chosen values v' the values v'', β'' and the corresponding impact stresses σ' and σ'' are determined and plotted versus v' (see Figure 43). The intersection of both curves yields $\sigma' = \sigma''$ for β_1'.

FIGURE 43



For other values β_2', β_3', .. in the same way the points $\sigma' = \sigma''$ are obtained. The curve $\sigma' = \sigma''$ determines β' as functions of V'. For the corresponding values of this function the kink angles $\theta_1 - \theta'$ and θ'' are determined and θ' and θ'' plotted versus V'. The intersection of both curves yields $\theta' = \theta'' = \theta$ and the corresponding value $\sigma' = \sigma'' = \sigma$ representing the stress value propagating over the kink.

At practically occurring impact speeds and impact angles the kink angle θ_1 is usually small, say less than 20°. A longitudinal stress wave arriving at such kink is, therefore, comparable with an oblique impact under an angle β which is less than 20°. Figure 32b shows that such impact results in a stress which is nearly equal to the stress produced at the impact angle $\beta = 0$. This means that <u>a stress value propagates over a kink with small kink angle (say less than 20°) approximately without disturbance</u>. This is in accordance with experimental observations and can be proved exactly by evaluation using the correct method described before. This result considerably simplifies the transient stress analysis, for instance, in an aircraft arresting gear cable.

12. Cable Impact at a Sheave

An infinitely long cable is guided over a sheave S as shown in Figure 44 and is subject to a transverse impact with the velocity V_o at its endpoint O, the initial stress being σ_o. A kink wave with the angle Θ_1 forms and propagates toward S. The impact stress σ_1 runs ahead of the kink wave and passes over the sheave S before this wave reaches S. The stress σ_1 produces a particle velocity u_1 ahead of the kink wave which will turn the sheave in clockwise direction with the circumferential velocity u_1. The last statement involves that the sheave has no mass. However, we will assume that also in the case of a sheave with mass the sheave turns with the velocity u_1 for the purpose of the following investigation. In this case we can assume the required rotation of the sheave has been produced by a rotational impact at the sheave in the moment where the stress wave passed over it.

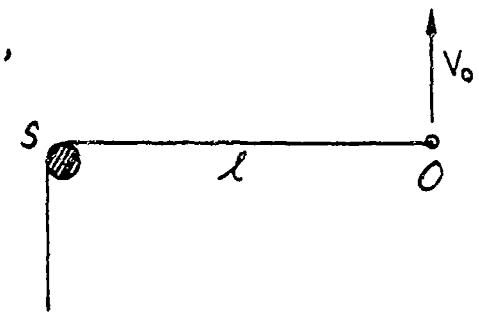


FIGURE 44

When the kink wave reaches the sheave, it is disturbed due to the fact that it cannot propagate anymore toward the left as before. We will show here that this disturbance is equivalent to a certain oblique impact from which the stress after the disturbance can be determined.*

In Figure 45 the shape of the cable after the impact of the transverse wave at the sheave S is indicated by the points S, Q and P. If there would have been no sheave the cable segment SP' would have continued its motion with the velocity V_o into the position RP during time t. The presence of the sheave at S over which the cable is

* Compare reference 9.

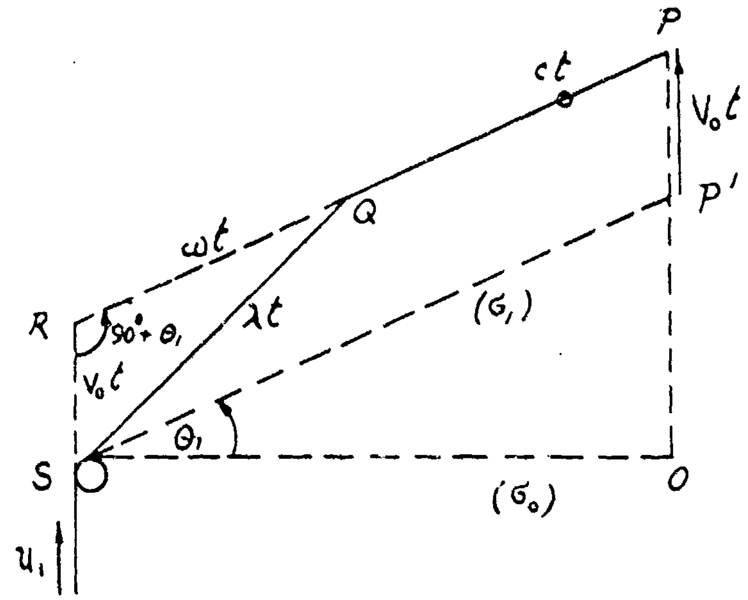


FIGURE 45

bended downward results in the deformation SQ of that segment. The length λt of SQ is related to V_0, θ_1 and the kink velocity ω with which Q moves toward P by

$$\lambda^2 = V_0^2 + \omega^2 - 2 V_0 \omega \cos(90° + \theta_1).$$

The elongation εt which the segment SP' experiences during time t is determined by

$$\varepsilon = \lambda - \omega - u_1$$

because cable is fed over the sheave with the velocity u_1 due to the initial impact at the cable. According to Hooke's law, therefore,

$$\frac{\frac{\sigma - \sigma_1}{E}}{1 + \frac{\sigma_1}{E}} = \sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\frac{V_0}{c}\frac{\omega}{c}\cos(90° - \theta_1)} - \frac{\omega}{c} - \frac{u_1}{c}. \qquad (191)$$

It is assumed that the longitudinal stress wave does not extend up to point P.

If we use approximately $\omega = \bar{c}$ (see formulas (157) and (165)) and neglect $\frac{\sigma_1}{E}$ compared with unity we obtain

$$\frac{\sigma - \sigma_1}{E} = \sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\frac{V_0}{c}\frac{\omega}{c}\cos(90° - \theta_1)} - \frac{\omega}{c} - \frac{u_1}{c}$$

where now

$$\frac{\omega}{c} = \frac{\bar{c}}{c} = \sqrt{\frac{\sigma}{E}}$$

FIGURE 46
KINK ANGLE AT TRANSVERSE IMPACT

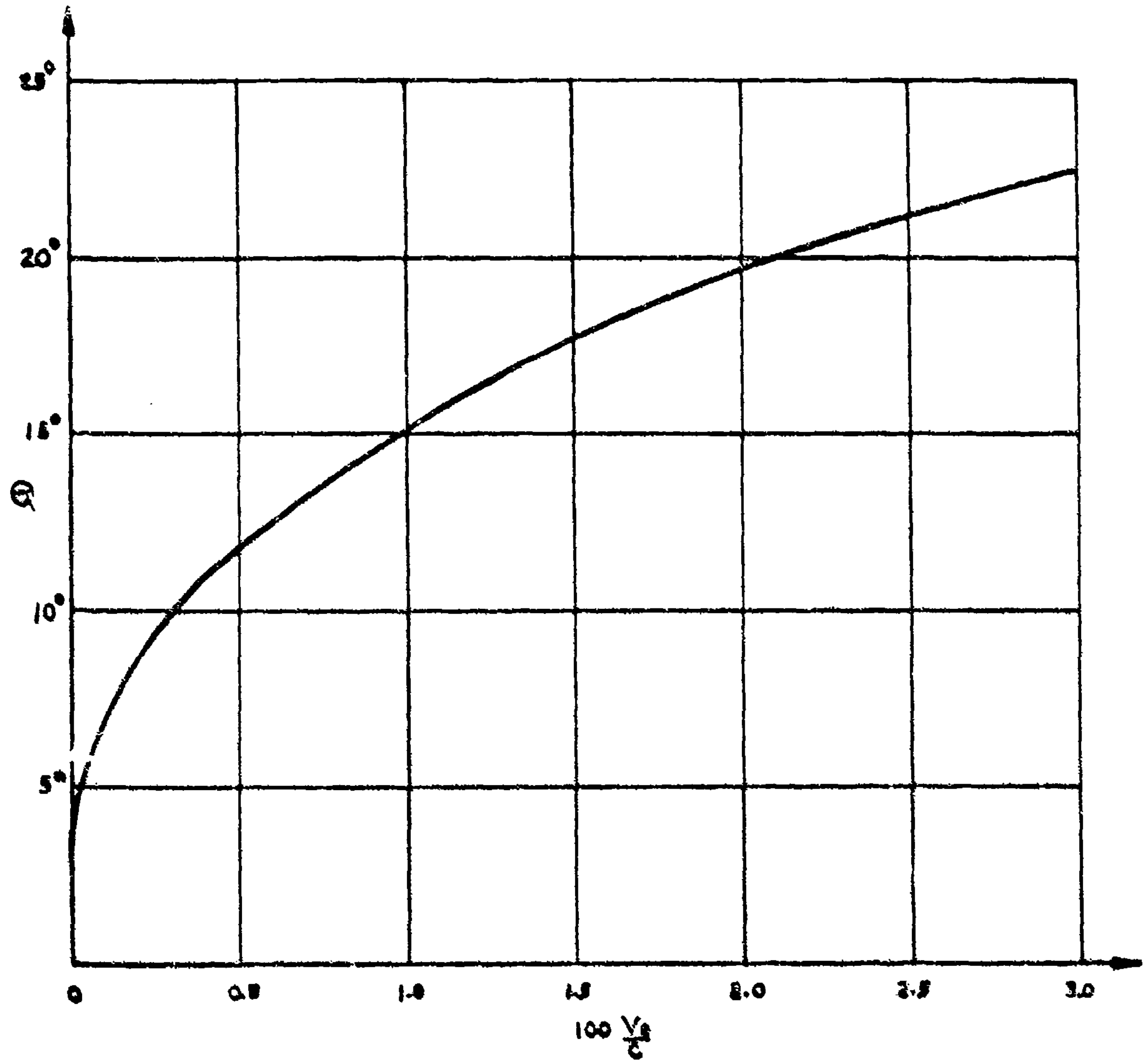
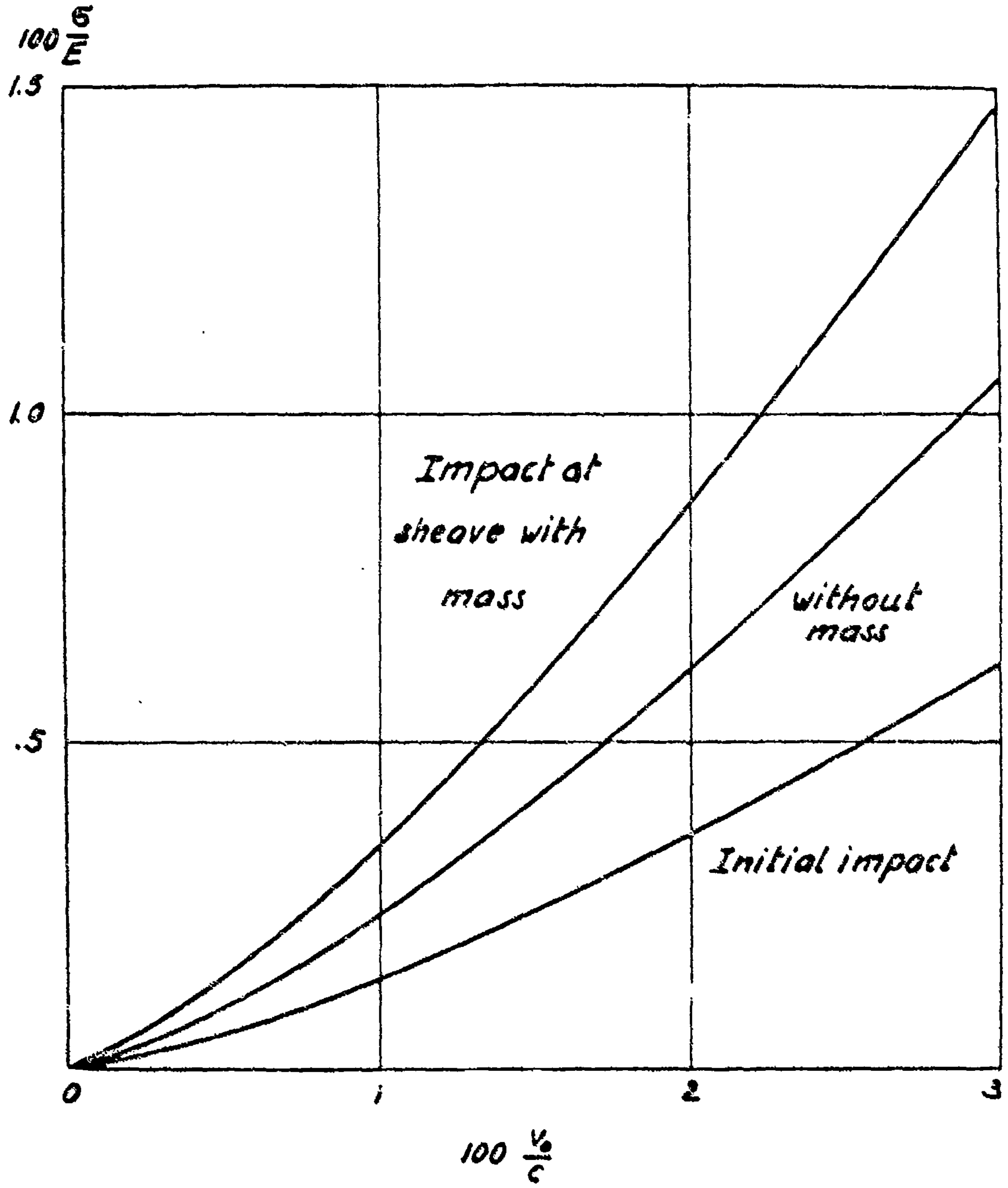


FIGURE 47

SIMPLE IMPACT OF A CABLE AT A SHEAVE



is independent of σ_1. Since

$$\frac{\sigma_1 - \sigma_0}{E} = \frac{u_1}{c}$$

we get

$$\frac{\sigma - \sigma_0}{E} = \sqrt{\left(\frac{V_0}{c}\right)^2 + \left(\frac{\omega}{c}\right)^2 + 2\frac{V_0}{c}\frac{\omega}{c}\cos(90^\circ - \theta_1)} - \frac{\omega}{c} \qquad (182)$$

Compared with formula (158) this result shows that the stress σ due to impact of the transverse wave at the sheave is approximately the same as the stress produced by oblique impact at a cable with the initial stress σ_0 under the impact angle $\beta = 90^\circ - \theta_1$ with the velocity V_0.

This result is represented by figures 46 and 47 where the first figure shows the angle θ and the second the values of $\frac{\sigma}{E}$ in comparison with $\frac{\sigma_1}{E}$ for an initial stress $\sigma_0 = 0$. The values of $\frac{\sigma}{E}$ prove to be approximately the same as the values obtained if a cable with the initial stress σ_1 is subject to an oblique impact under the angle $\beta = 90^\circ - \theta_1$ with the impact velocity $V_0 - u_1$.

It has to be noticed that the computed stress value holds only for a short time after the impact. The stress difference on both sides of the sheave will start to accelerate the sheave, a process which can be computed according to section III 1. On the other hand, the stress difference $\sigma - \sigma_1$ will propagate toward point P and generally be reflected at this point. The returning longitudinal wave will produce a disturbance of the kink at Q as discussed in section IV 11 and will change the motion of the sheave and the stress in the cable in the neighborhood of the sheave once more.

If the sheave is assumed to be massless or if it is assumed that there is no friction between sheave and cable, the stress due to the impact of the transverse wave at the sheave spreads over both sides of the cable adjacent to the sheave. Thus, the influenced cable length at the time t after the impact is equal $2ct$ instead of ct. Instead of formula (191) we obtain in this case the formula

$$\frac{\frac{\sigma-\sigma_1}{E}}{1+\frac{\sigma_1}{E}} = \frac{1}{2}\left[\sqrt{\left(\frac{v_0}{c}\right)^2+\left(\frac{\omega}{c}\right)^2+2\frac{v_0}{c}\frac{\omega}{c}\cos(90^\circ-\theta_1)}-\frac{\omega}{c}-\frac{v_1}{c}\right] \tag{193}$$

Using the same approximation as before instead of formula (192) the formula

$$\frac{2\sigma-(\sigma_1+\sigma_0)}{E} = \sqrt{\left(\frac{v_0}{c}\right)^2+\left(\frac{\omega}{c}\right)^2+2\frac{v_0}{c}\frac{\omega}{c}\cos(90^\circ-\theta_1)} - \frac{\omega}{c} \tag{194}$$

is obtained where now

$$\frac{\omega}{c} = \sqrt{\frac{\sigma}{E}}\,.$$

The values of $\frac{\sigma}{E}$ following from formula (194) are plotted for $\sigma_0 = 0$ in Figure 47.

CHAPTER V: DYNAMICS OF AN AIRCRAFT ARRESTING GEAR CABLE

1. Scheme of the Mark 5 and the Mark 7 Arresting Gears

In this chapter the general results of the preceding chapters are applied to the cable problems arising with the development of aircraft arresting gears.* We start with arresting gears of the Mark 5 or Mark 7 type which are - with respect to the use of the cable - essentially not different. Figure 48 shows the Mark 5 arresting gear schematically. A steel wire cable with hemp core, the deck pendant, is stretched across the deck of the aircraft carrier and is connected with two other such cables, the purchase cables, by means of terminals or links. They are reeved over the sheaves of a movable crosshead and a set of fixed sheaves as shown in Figure 48 and finally anchored. The movable crosshead is connected with a piston which acts against a hydraulic fluid in a cylinder, forcing this fluid through the variable orifice of a control valve into an accumulator, if the tailhook of the airplane engages the deck pendant and pulls it out. Thus the piston is pressed against the hydraulic fluid by nothing else than a common pulley.

In the scheme of Figure 48, the arrangement of the cable on the starboard and the port sides are completely symmetrical. If we assume that the tailhook of the airplane would engage the deck pendant perpendicularly to and exactly in the center of the deck pendant, no difference in the mechanics of the system on both sides would exist. The two anchored cable ends could be connected to each other in this case over another sheave without change of the mechanics of the system. Any system where the cable ends are connected over a sheave instead of being anchored is called an endless reeved system. However, only if the system is symmetrical to the center line, the anchored system and the endless reeved system are mechanically equivalent. At shipboard, the segments of purchase cable which are leading to the first engine sheaves are usually not equal in

* Compare reference 14.

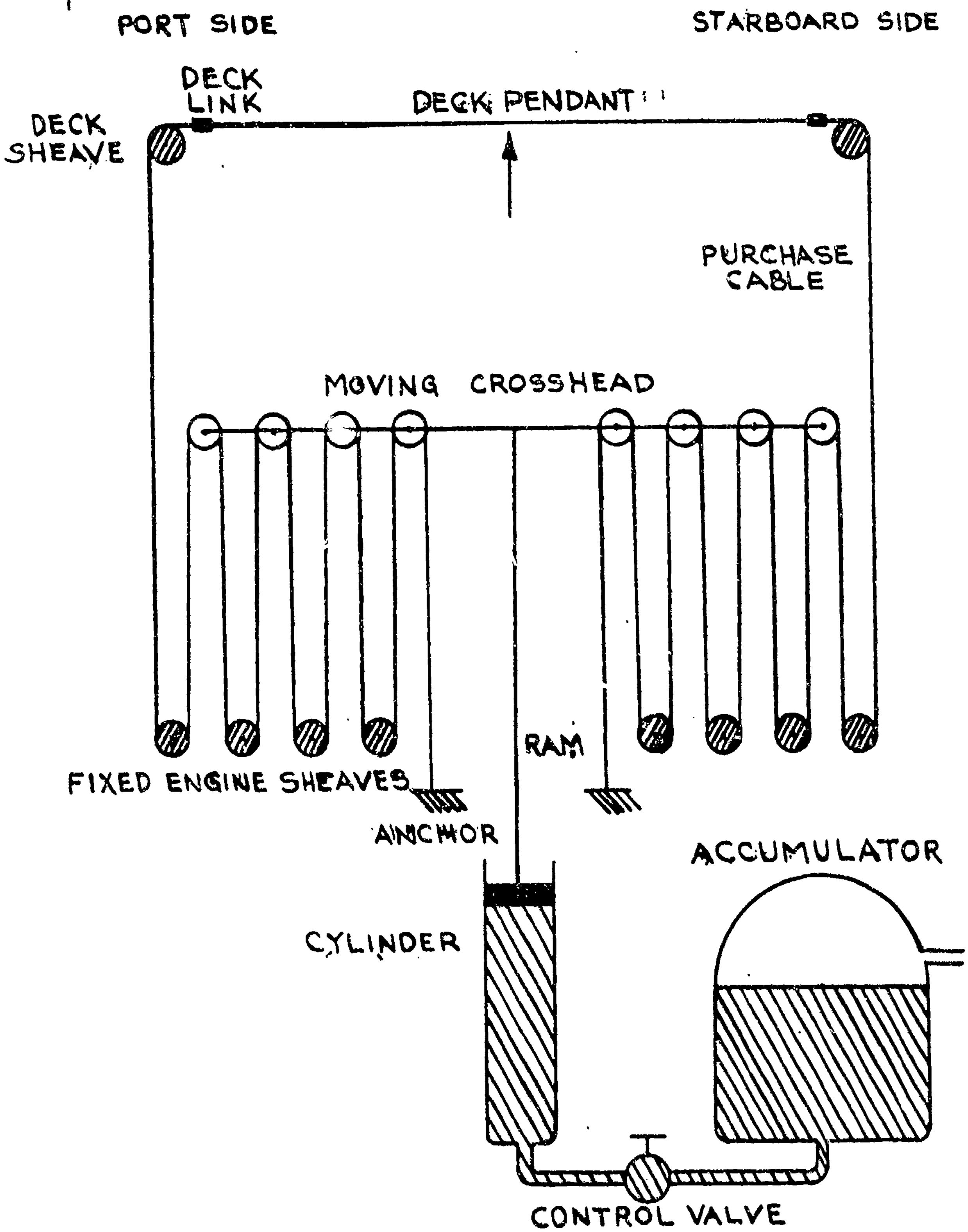

PORT SIDE
STARBOARD SIDE
DECK LINK
DECK PENDANT
DECK SHEAVE
PURCHASE CABLE
MOVING CROSSHEAD
RAM
FIXED ENGINE SHEAVES
ANCHOR
ACCUMULATOR
CYLINDER
CONTROL VALVE

FIGURE 48

length resulting in an unsymmetry of the total system beside the fact that the airplane engages the cable more or less off center. Only the arrangement in the engine will be symmetrical. Therefore, the anchored system and the endless reeved system will behave differently.

The number 2n of cable segments which are pulling at the sheaves of the movable crosshead on one side of the engine is called the reeving ratio. In the case of Figure 48, the reeving ratio 2n = 10. If T is uniformly the tension in each of these cable segments, the moving force acting on the piston is equal 4n T minus the force acting against the piston due to the pressure in the fluid.

The pressure of the fluid in the cylinder is automatically controlled by a valve. In the older types of Mark 5 and Mark 7 arresting engines, the orifice opening of the valve was controlled by constant pressure air acting on a spindle while in recent types of Mark 7 arresting gear the orifice opening is controlled by the stroke of the engine using a cam. The general goal of any automatic control device is to produce a force F against the piston so that a desired deceleration of the airplane mass is obtained either as function of stroke or time. The basic problem is, therefore, to determine the required force F which results in a prescribed deceleration. The control device by which the determined force F can be obtained is a fluid flow problem not to be discussed in the present monography. We assume here that any required force F as function of stroke or time can be produced.

In the following, arresting gears of a type as described before or equivalent to such will be called conventional. We do not restrict, however, the discussion to conventional types but will consider as well variants and completely different designs.

2. Impact at the Deck Pendant

The simplest arresting gear using a cable would consist in an infinitely long stretched cable only (see Figure 49).

This idealization is of interest because it shows the upper limit for the velocity v_o of an airplane which possibly can be arrested by a cable.

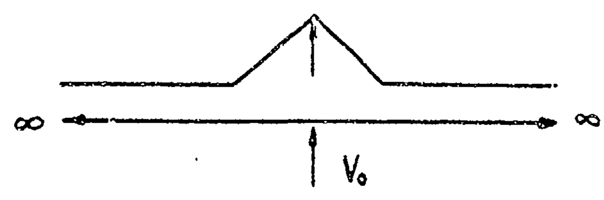


FIGURE 49

If we assume that the initial stress of the cable is σ_o and that the hook of the airplane engages the cable perpendicularly with the velocity v_o the impact stress is determined by the transverse impact formula derived in Sections IV 5 and 6 and represented graphically by Figure 33. This figure shows that the smallest impact stress is obtained in any case if $\sigma_o = 0$. Table 1 in Section I 5 contains the breaking strengths for steel wire cables with hemp core as commonly used in arresting gears. If we divide the breaking strengths by the corresponding metallic cross section areas we obtain the breaking stresses σ_{max} as shown in Figure 50 for cable diameters between 11/16 and 2 inches. The breaking stress decreases somewhat with increasing diameter but does not differ much from the constant value $\sigma_{max} = 0.335 \cdot 10^8$ lbs/ft². Thus approximately

$$100 \frac{\sigma_{max}}{E} = 1.83$$

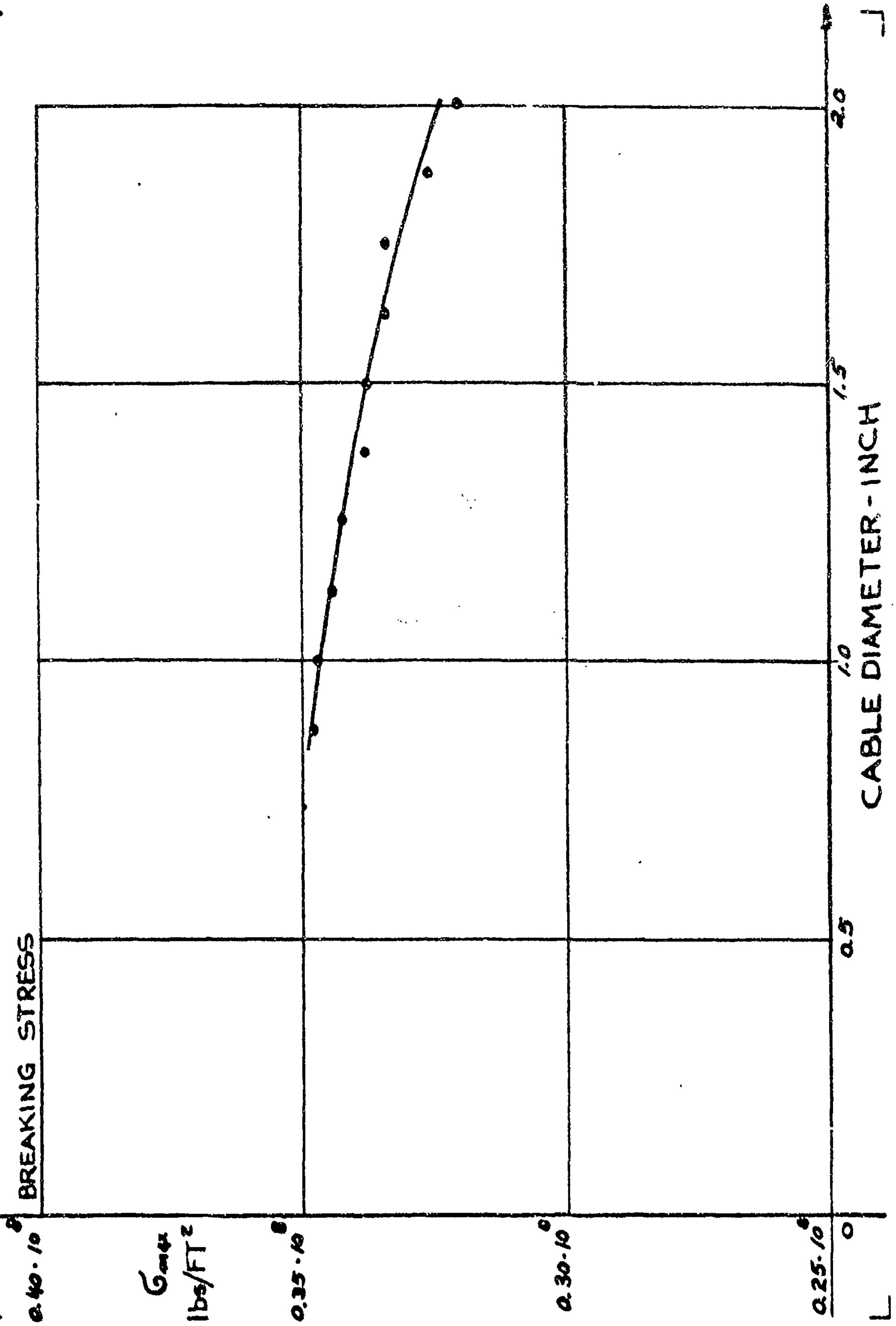
for all steel wire cables with hemp core if an average elasticity modulus of $E = 18.3 \; 10^8$ lbs/ft² is assumed. From Figure 33 follows that this ratio corresponds to the value

$$100 \frac{V_o}{c} = 6.80$$

for the impact velocity v_o at $\sigma_o = 0$. Because $c = 10020$ ft/sec is the longitudinal wave velocity, therefore

$$v_o = 678 \text{ ft/sec} = 402 \text{ kts}$$

FIGURE 50


BREAKING STRESS
CABLE DIAMETER - INCH
0.5
1.0
1.5
2.0
0
$0.40 \cdot 10^8$
σ_{max} lbs/FT^2
$0.35 \cdot 10^8$
$0.30 \cdot 10^8$
$0.25 \cdot 10^8$

This value represents the upper limit for the velocity of an airplane which possibly can be arrested using a steel cable with hemp core of conventional construction.

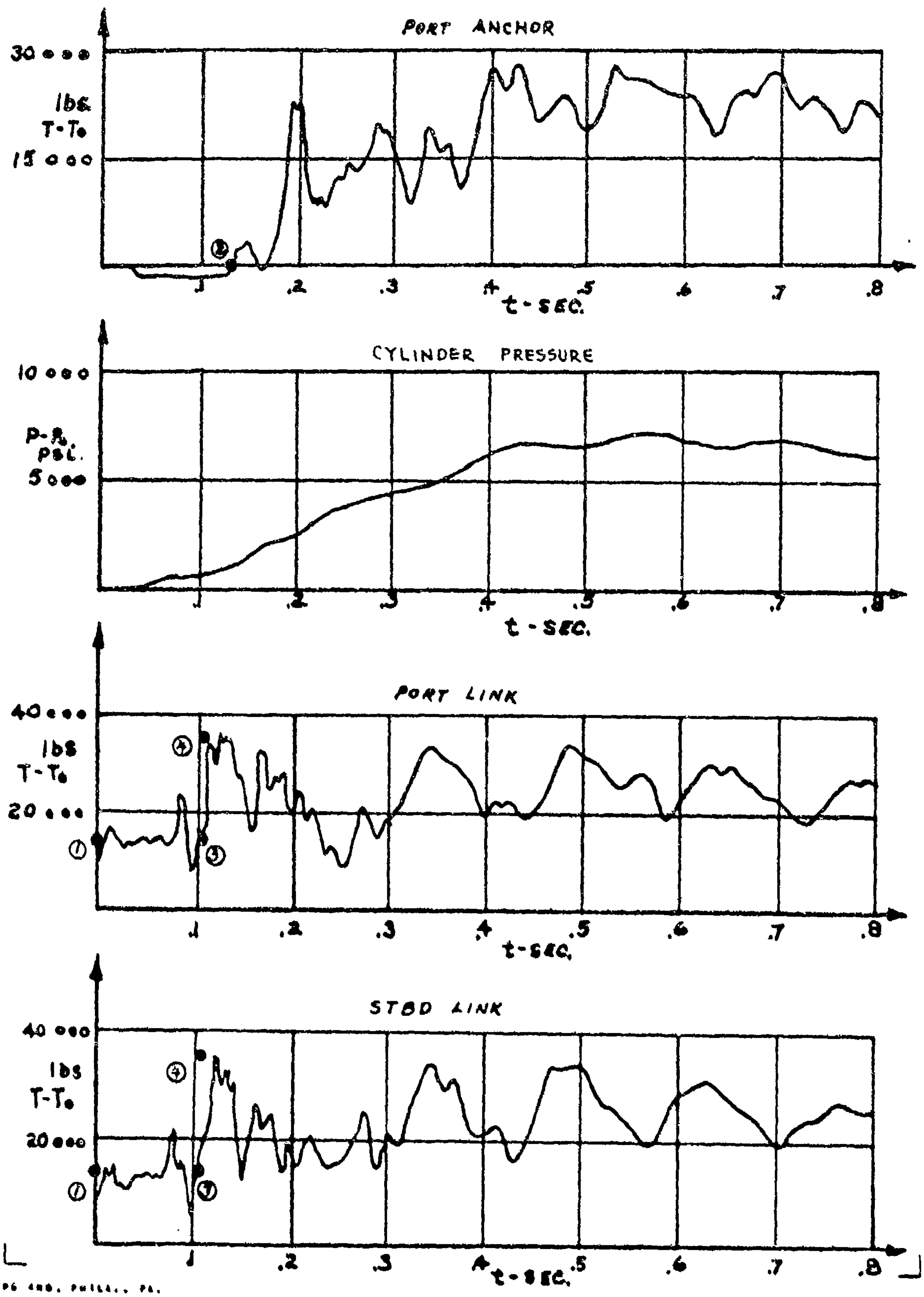
This limit is independent of the cable size. It depends only on the breaking strength σ_{max}, the elasticity modulus E and the longitudinal wave velocity c. If a safety factor of 60% is required which is a common requirement for aircraft arresting gears, the upper limit for the engaging velocity becomes

$$v_o = 241 \text{ kts.}$$

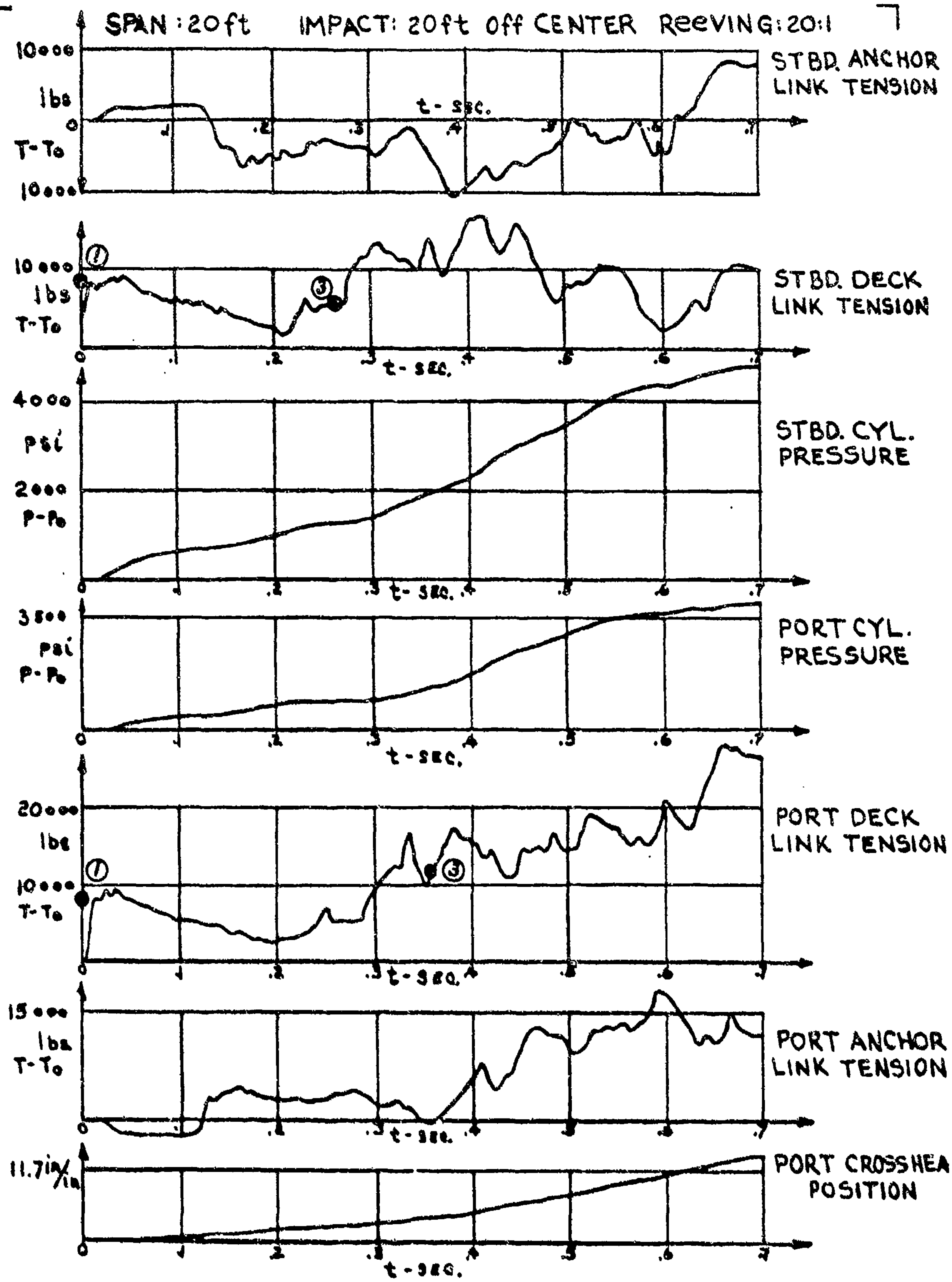
For a very short time any kind of arresting gear using a cable as arresting device behaves like a cable of infinite length. If l is the length of the half span of the deck pendant of any such arresting gear, the stress produced by the impact in the center of the deck pendant requires the time $\frac{l}{c}$ to travel to the deck sheaves. During this time, the arresting engine or any device between the deck sheave and the engine is completely insensitive to what happened at the point of engagement of the cable. This time is in general very short. If, for instance, the length of the deck pendant is $2l = 100$ ft the time in question is $\frac{l}{c} = 0.005$ sec for any size of a conventional cable. However, this consideration proves that the stress due to the perpendicular impact of the airplane hook at the cable depends on the airplane velocity v_o the cable material constants E and c and its initial stress σ_o only and is independent of the type of arresting gear, the weight of the airplane and the diameter of the cable.

This stress is determined by the transverse impact formula as derived before and represented graphically by Figure 33 for any given data. In Figure 35 the resulting impact tensions are plotted for three frequently used types of cable. Figure 34 shows a comparison of the theoretical impact stress with

FIGURE 51


PORT ANCHOR
30 000
lbs.
T-T₀
15 000
.1
.2
.3
.4
.5
.6
.7
.8
t-SEC.
CYLINDER PRESSURE
10 000
P-P₀
PSI.
5 000
t-SEC.
PORT LINK
40 000
lbs
T-T₀
20 000
t-SEC.
STBD LINK
40 000
lbs
T-T₀
20 000
t-SEC.

SPAN : 20ft IMPACT: 20ft Off CENTER REEVING: 20:1



V_o = 112.6 ft/sec. W = 4000 lbs T_o = 1800 lbs 1" cable measured values obtained from different arresting gears and cable sizes.

Figure 51 represents a typical cable tension measurement from a Mark 5 arresting gear. The cable diameter d = 1". The impact velocity v_0 = 101.5 kts. Point (1) represents the theoretical impact tension. The tensions T are measured in the deck links on the starboard and the port sides near the deck sheaves. The initial tension T_0 = 1200 lbs. The engagement occurred in the center of the deck pendant.

In the case of an off-center engagement, the impact stresses measured in the starboard and the port link must be the same. There is, however, a time difference because the stress has to travel on one side over a longer cable length than on the other side. This is demonstrated in Figure 52 representing an off-center engagement at a 240 ft. deck pendant. The points marked by (1) are again those corresponding to the theoretical impact tension. The engagement is 20 ft. off-center toward starboard. Accordingly, the stress arrives at the starboard link at first and is registered at the port link 0.004 seconds later as can be seen clearly in the oscillogram. This measurement has been done at an arrangement where each cable end is reeved separately over a Mark 5 arresting engine so that the arresting gear contains two arresting engines. Accordingly, the measurement shows two cylinder pressure curves.

There is the possibility that the engagement of the deck pendant is not a perpendicular one. If in such caseof oblique impact the cable does not slip over the hook the influence of only a few degrees difference from the perpendicular engagement is considerable. From Figure 32 can be seen, for instance, that at a value of $100 \frac{v_0}{c} = 2$ the impact stress at $\beta = 85°$ is given by $100 \frac{\sigma}{E} = 0.5$ (at zero initial stress) while at $\beta = 95°$ the impact stress is given by $100 \frac{\sigma}{E} = 0.27$. Therefore, if an engagement would occur under an angle of only 5° against the perpendicular direction the stress on one side of the hook would be nearly twice of that on the other side (See Figure 53).

Such large stress difference will in general be balanced by slippage of the cable over the hook. At smaller deviations from the perpendicular direction, however, considerable stress differences in the cable on both sides of the hook can occur. Actually the impact stresses measured in both deck links are seldom identical. The difference can mostly be explained by a small deviation from the perpendicular impact direction.

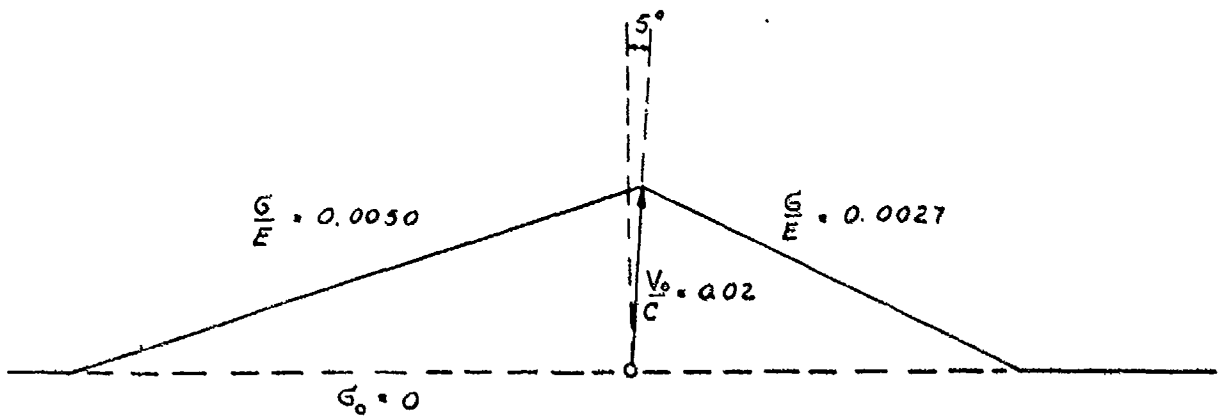


FIGURE 53

3. Slack Due to Crosshead Motion

The stress due to impact at the deck pendant propagates with the velocity c (~10000 ft/sec for a steel cable with hemp core) and reaches the first sheave of the engine crosshead (see Figure 48) after a time t, which depends on the length of the cable between the point of impact and the first engine sheave in question. At this time the crosshead will start its motion. In the case of a symmetrical arresting gear and center landing, both first crosshead sheaves on starboard and port side will be under tension at the same time t. If T_1 is the impact tension, then at the time t_1 the force 4 T_1 will start to move the crosshead against the initial cylinder pressure. Up to this time the tension in the deck pendant will be approximately constant equal T_1 because the force acting on the airplane during that short time will not reduce the speed appreciably. Point (2) in Figure 51 indicates the time where the stress wave arrives at the first crosshead sheave and up to which the tension is constant equal T_1.

The acceleration of the crosshead is usually large enough that it obtains a considerable speed before the impact stress wave reaches the anchor or even the second or third crosshead sheave. This means that toward the anchor end of the cable the anyhow low initial tension of the cable will drop immediately to zero. Thus slack forms in the cable. No appreciable stress propagates over a cable in slack condition as has been shown in Section IV 5. Thus no stress will form at the anchor and until all slack has been picked up which happens with the particle velocity of the cable, only. If all slack has been picked up, the anchor suddenly stops the particle velocity of the cable producing a longitudinal impact tension which propagates in opposite direction.

A quantitative estimation of the slack formation due to crosshead motion can be obtained in the following way (see Figure 54). We assume that the reeving ratio is 2n (= 10 in the figure) and that at the time t on each of both sides of the engine 2ν cables are under tension. We further assume that the force on the crosshead due to the cylinder pressure increases with time t linearly

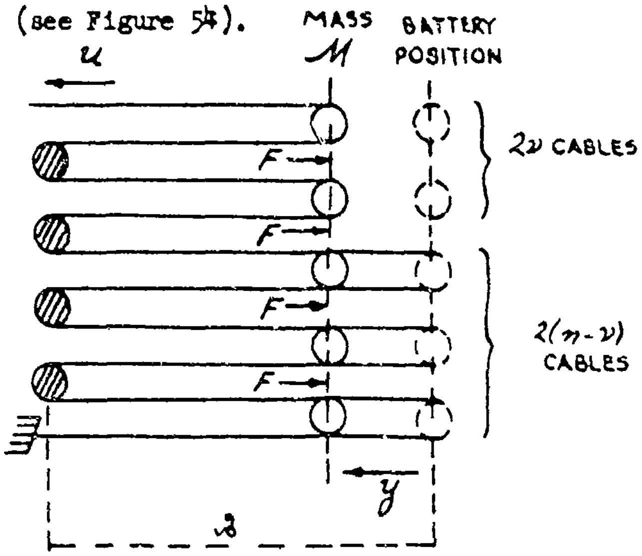


FIGURE 54

$$F = F_1 t \tag{195}$$

where F_1 is a constant, from the moment where the free cable end begins to move longitudinally with a given constant velocity u. The zero point for the time is the moment where F begins to increase. The equation of motion of the crosshead with the mass M including sheaves and ram is

$$M\ddot{y} = 4\nu(T - T_0) - F_1 t \tag{196}$$

where T is the average tension in the 2ν cables on each side which are pulling at the crosshead. ν is increasing with the time discontinuously in unity steps. For mathematical convenience, however, this stepwise increase is replaced by a continuous increase from $\nu = 0$ to $\nu = n$. If $\nu = n$ the slack has been picked up completely. We denote the time for which $\nu = n$ by t^*. The problem to be solved is to determine this time t^*.

According to Hooke's law at a time $t \leq t^*$

$$\frac{ut - 2\nu y}{2\nu\lambda} = \frac{T - T_0}{qE} \tag{197}$$

because $ut - 2\nu y$ is the elongation of the cable under tension T at this time while $2\nu\lambda$ is the original length of the elongated cable segment. The time t needed for building up the tension T within the 2ν cables

$$t = \frac{2\nu(\lambda - y)}{c} + \frac{2\nu y}{u} \tag{198}$$

because the stress propagates in a cable under tension with the longitudinal wave velocity c and is stopped until slack is picked up with the velocity u approximately. Thus

$$2\nu = \frac{t}{\frac{\lambda - y}{c} + \frac{y}{u}} \tag{199}$$

and because of (197)

$$T - T_0 = qE\,\frac{u}{\lambda}\,\frac{\lambda - y}{c}. \tag{200}$$

Therefore, equation (196) takes the form

$$\ddot{y} = \left(\frac{2qEu}{\lambda M}\,\frac{\frac{\lambda - y}{c}}{\frac{\lambda - y}{c} + \frac{y}{u}} - \frac{F_1}{M}\right)t. \tag{201}$$

If we neglect on the right side of this equation y as small compared with λ we obtain

$$\ddot{y} = \left(\frac{2qEu}{M}\,\frac{1}{\lambda + \frac{c}{u}y} - \frac{F_1}{M_1}\right)t. \tag{202}$$

This differential equation shows that $\ddot{y} = 0$ for $t = 0$

$$y = \frac{u}{c}\left(\frac{2qEu}{F_1} - \lambda\right) \tag{203}$$

We can expect that the slack is picked up in the moment where the crosshead reaches the highest acceleration. We can further assume that the maximum $\ddot{y}$ is approximately situated in the point $\frac{y}{2}$ where y is the value given by equation (203). We denote this value by y* and get

$$y^* = \frac{u}{c}\left(\frac{qEu}{F_1} - \frac{\lambda}{2}\right)$$

This is the y-value corresponding to t* which follows from (198) for $\nu = n$ and is approximately

$$t^* = \frac{2n\lambda}{c} + \frac{2ny^*}{u}$$

Substituting here the value for y* we obtain

$$t^* = \frac{2n}{c}\left(\frac{qEu}{F_1} + \frac{\lambda}{2}\right) \tag{204}$$

<u>representing the time at which the slack due to the crosshead motion has been picked up. Here 2n is the reeving ratio, u the velocity of the cable, E its elasticity modulus, c its longitudinal wave velocity, q its metallic cross section area, λ the distance between crosshead sheaves and fixed engine sheaves in battery position and F_1 the slope versus time of the force acting on the crosshead due to the cylinder pressure.</u>

This consideration has to be justified, of course, by the actual solution of the non linear differential equation (202) which can be obtained by a power series of the form $y = \alpha_3 t^3 + \alpha_6 t^6 + \cdots$

<u>Example</u>: In the case of the Mark 5 arresting gear, the following data are valid: $E = 18.3 \cdot 10^8$ lbs/ft^2, c = 10,000 ft/sec, q = 0.00274 ft^2 for its 1" - diameter cable and $\lambda = 23$ ft. We consider the case of a 2n = 12 reeving ratio and an engaging velocity of $v_o = 171.3$ ft/sec for which Figure 51 shows measured values. The impact stress σ is given by

$$\frac{\sigma - \sigma_o}{E} = \frac{u}{c} = 0.00281$$

(see the graph Figure 32b). From the cylinder pressure curve of Figure 51 we obtain the force F acting on the crosshead by multiplying the pressure p with the piston area a = 86.6 in.^{2} and a friction factor of 1.1. The slope is found to be

$$F_1 = 1.463.10^6 \text{ lbs/sec}$$

Now

$$\frac{q E u}{F_1} = \frac{0.00274.18^3.10^2.28.1}{1.463} = 96.3$$

and

$$t^* = \frac{12}{10000} (96.3 + 11.5)$$

or

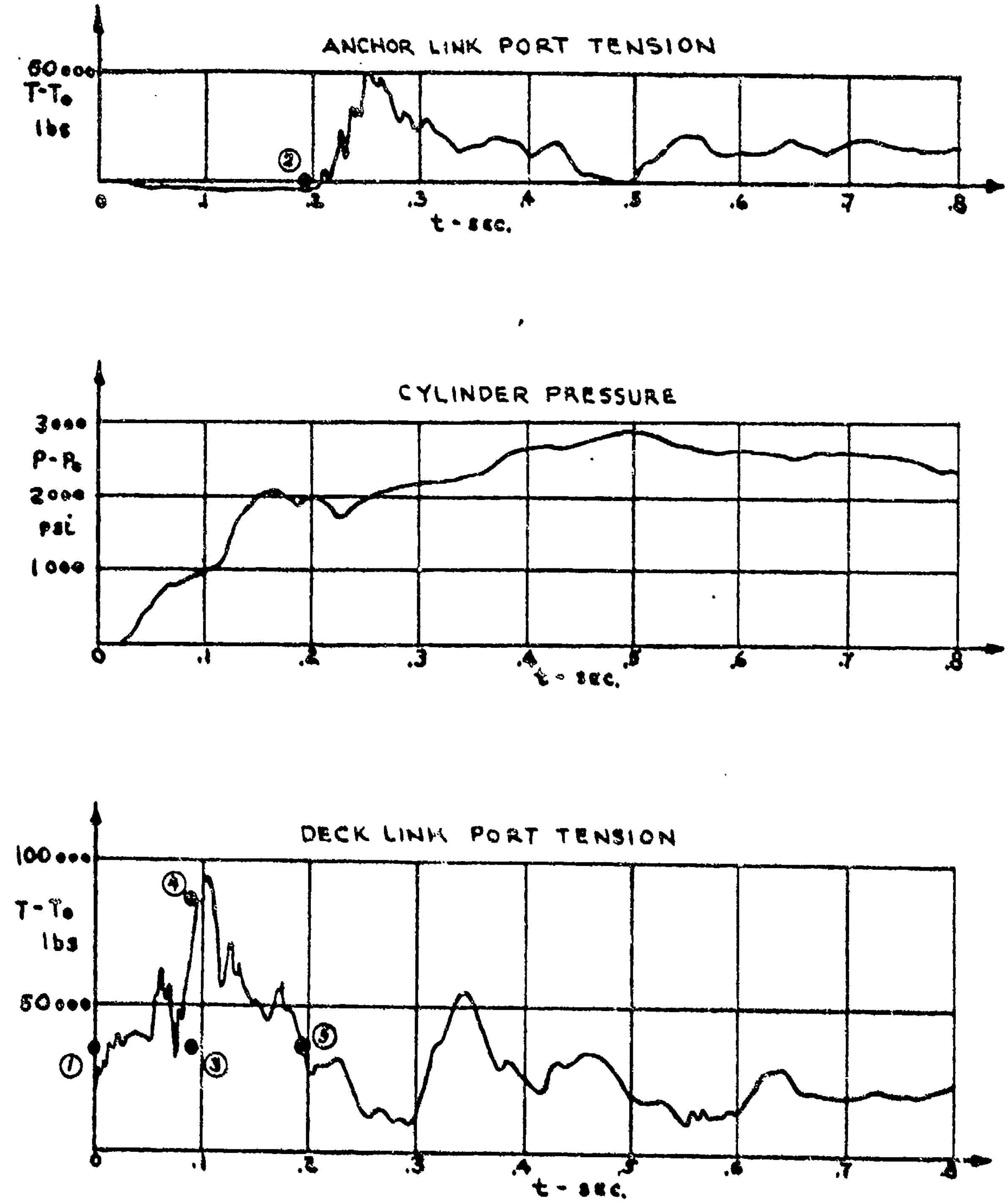
$$t^* = 0.128 \text{ sec.}$$

The moment from which the time t* has to be counted is the moment at which the impact stress reached the distance λ before the first crosshead sheave. Between decklink and this point, a cable length of 43 feet was located. Thus, the time in Figure 51 corresponding to t* is equal t* + 0.0043 = 0.132 sec. The point indicating this time is denoted by 2. The anchor tension curve shows that at this time the slack has actually been picked up completely.

Figure 55 shows measured values in the case of a Mark 7 arresting gear where $q = 0.00519$ ft^2, $\lambda = 35.5$, $2n = 18$, $v_o = 208$ ft/sec and $F_1 = 4.53.10^6$ while the other data are the same as before. In this case, formula (204) yields t* = 0.172 sec. Point 2 indicates this value. Other measurements even under very different conditions show good agreement with this theory too, though the particle velocity u computed from the impact stress and used as constant value in these computations can be considered only as a rather crude approximation for the actually variable velocity u with which the cable is moving.

It is remarkable that the mass M of the crosshead does not explicitly appear in formula (204). This does not mean, however, that the result is independent of M because the constant F_1 depends on M.

FIGURE 55


ANCHOR LINK PORT TENSION
60 000
T-T₀
lbs
0
.1
.2
.3
.4
.5
.6
.7
.8
t - sec.
CYLINDER PRESSURE
3000
P-P₀
2000
psi
1000
t - sec.
DECK LINK PORT TENSION
100 000
T-T₀
lbs
50 000
t - sec.

The result expressed by formula (204) shows that the slack increases with increasing reeving ratio 2n, with increasing engaging velocity v_o (resulting in increasing u), with increasing cable diameter, with increasing λ (increasing length of the engine) and with decreasing slope F_1 of the pressure force on the crosshead.

The influence of λ, however, is relatively small compared with that of the reeving ratio 2n so that increasing the reeving ratio and decreasing the engine length does not appear as an adequate means for slack prevention. The main parameters, influencing slack formation, are the reeving ratio 2n, the particle velocity u and the constant F_1. For arresting gears with constant runout valve, however, the ratio $\frac{u}{F_1}$ is rather constant and the slack formation, therefore, independent of the engaging speed v_o.

Formula (200) shows that the tension T stays approximately constant during the process of slack pick up as long as u can be considered as approximately constant.

The time t* of complete slack removal is of importance because from this time on any stress disturbance will pass through the cable with the speed of sound c unless new slack is produced which is in general not the case. However, at the time t* the particle motion reaches the anchor and is stopped there which results in a tension increase as is to be seen in Figures 51 and 54. The increased force at the anchor side of the crosshead produces higher acceleration of it, and a reduction of the tension near the first crosshead sheave until the tension produced at the anchor has passed through the engine cables and arrived at the deck link. This mechanism is clearly to be seen in the measurement represented by Figure 52 where due to the large span the slack is picked up completely before any new stress disturbance at the deck pendant is produced. In the cases of the measurements represented by Figures 51 and 55, however, the transverse wave reaches the deck sheave shortly after respectively before the slack has been picked up completely so that a sudden tension increase occurs which superposes the decreasing tension due to the impact at the anchor.

4. Impact of the Cable at the Deck Sheave

At conventional arresting gears, the stress in the deck pendant can be considered as approximately constant until the transverse wave produced by the impact of the hook reaches the deck sheave. Subsequently, the moving cable impacts the sheave, resulting in a sudden increase in stress which can be computed from Section IV 12. Figure 47 showed the stresses produced by this impact for the two cases of a sheave with mass and a sheave without mass, the latter case corresponding to slipping of the cable over the sheave. Figure 47 had been computed for zero initial stress of the cable. For small initial stresses σ_0 in comparison with the resulting impact stresses, the initial stress σ_0 can be added to the results shown in Figure 47 in order to obtain the impact stresses at the initial stress σ_0. We obtain then from Figure 47 for any engaging velocity V_0 and any initial stress σ_0 the perpendicular initial impact stress σ_1 and the stress σ_2 due to impact of the transverse wave at the sheave.

The span of the deck pendant of a conventional arresting gear is usually small (less than 120 ft.) so that any stress wave propagates through it in a very short time. The stress increase $\sigma_2 - \sigma_1$ due to the cable impact at the sheave, therefore, will propagate in a very short time toward the engaging hook at P in Figure 56 and there be reflected completely. The returning stress amount $\sigma_2 - \sigma_1$ will pass over the kink in the cable. The angle θ_1 of the initial kink wave as shown in Figure 46 is small enough in the case of conventional engaging speeds in order to neglect the change of $\sigma_2 - \sigma_1$ while passing over the secondary kink in accordance with the investigation of Section IV 11. Thus, the stress in the deck pendant after the impact at the sheave is given by

$$\sigma = \sigma_2 + (\sigma_2 - \sigma_1) = 2\sigma_2 - \sigma_1$$

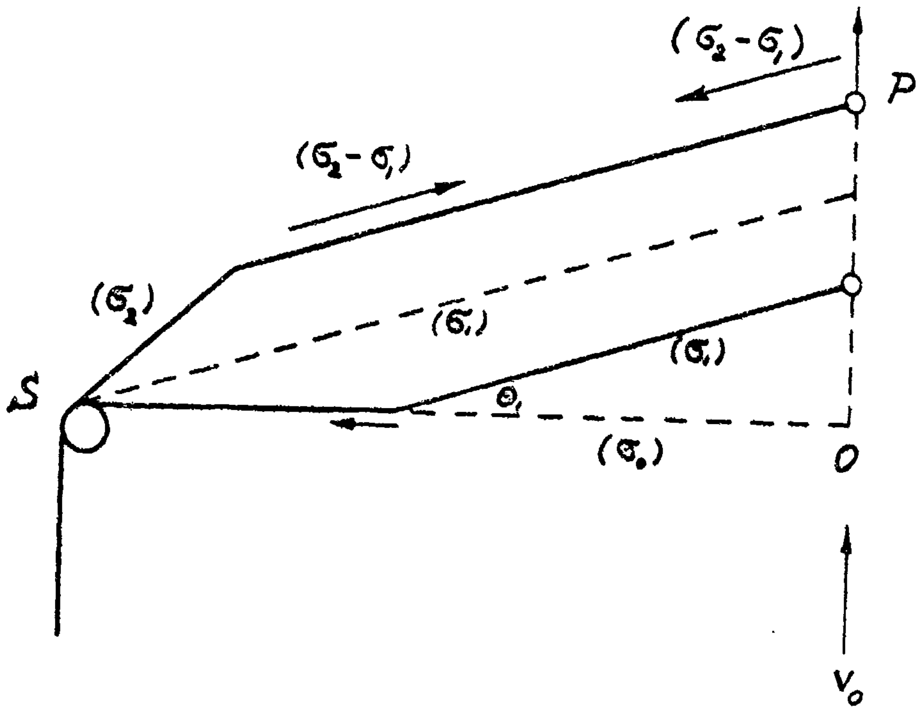

$(\sigma_2 - \sigma_1)$
P
$(\sigma_2 - \sigma_1)$
(σ_2)
(σ_1)
S
(σ_1)
θ_1
(σ_0)
O
V_0

FIGURE 56

Figure 57 shows these values versus the engaging speed in both cases of a sheave with and without mass (slipping). Also plotted is the initial impact stress σ_1 and for comparison the longitudinal impact stress. The theoretical curves are in accordance with measured values obtained from the Mark 5 and Mark 7 arresting gears, with 1 inch and 1-3/8 inch cable diameters, respectively. Most values for the stress after the impact at the sheave are scattered around a curve which would be situated very near to the curve corresponding to a slipping cable so that slipping more or less of the cable over the sheave must be concluded from this comparison of test and theory in spite of the high tensions under which it generally occurs.

The question whether or not the cable slips over the sheave has been answered only quite recently. Originally, slippage has been assumed as possible. Later, it has been rejected because the groove in the sheave never showed any signs of abrasion. On the contrary, it always showed the impressions of the cable wires. Recent high speed movie tests, however, decided the question in favor of slippage. They even showed that under tension the sheave rim can move faster than the cable. This result is in accordance with the observation that a stress wave in a cable in general experiences only negligably small disturbances when passing over a sheave.

The time t at which the impact of the cable at the sheave occurs is determined by the relation

$$t = \frac{\ell}{\omega}$$

where ℓ is the distance of the point of hook impact from the sheave (the half span in the case of a center landing) and ω the kink velocity which is given by formula (170)

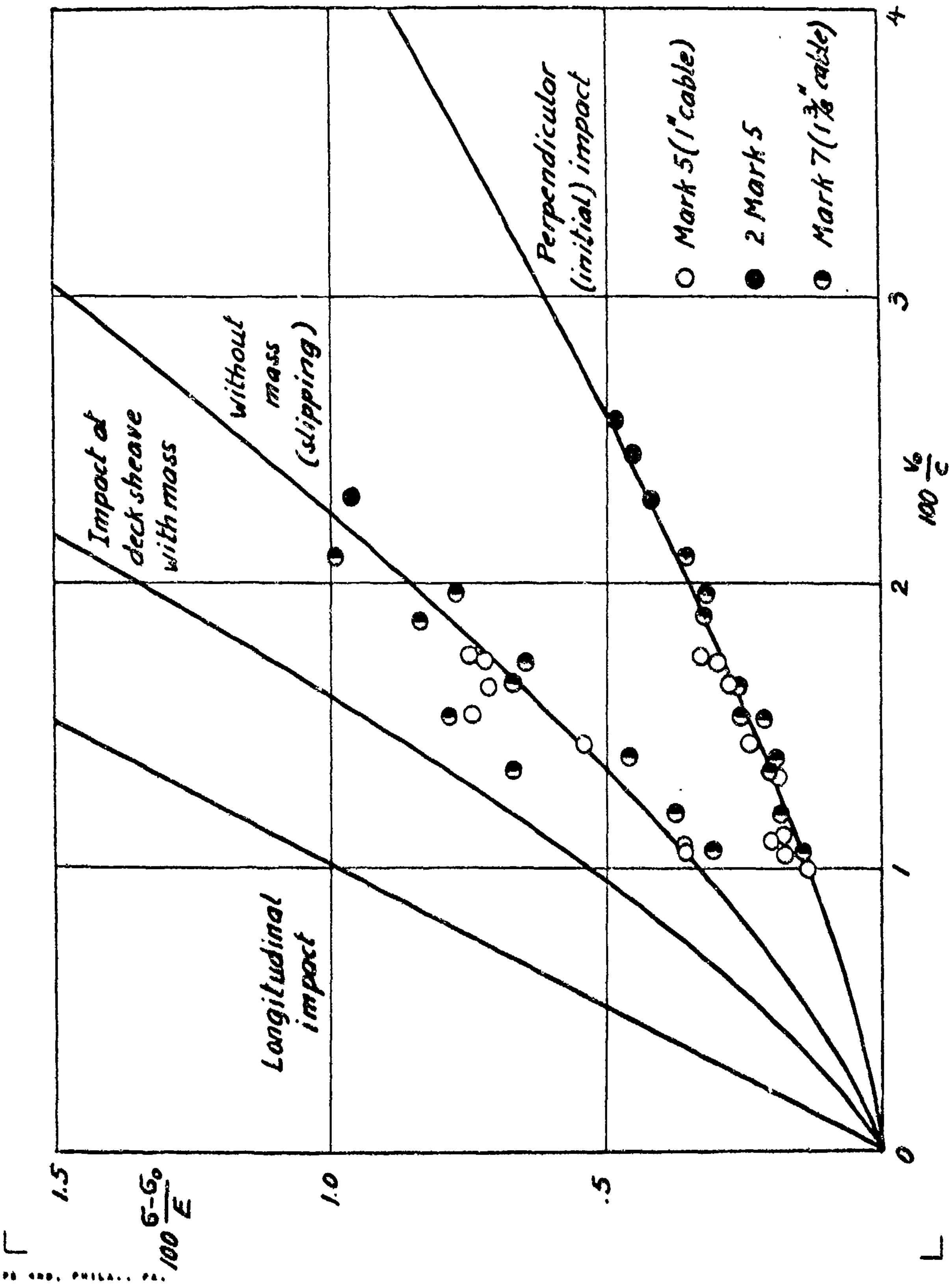
$$\frac{\omega}{c} = \sqrt{\frac{\sigma}{E}} - \frac{\sigma - \sigma_0}{E}.$$

$\frac{\omega}{c}$ is plotted in Figure 58 in the case of $\sigma_0 = 0$.

Point (3) in Figures 51 and 54 indicates this time.

FIGURE 57

CABLE IMPACT AT THE DECK SHEAVE OF AN ARRESTING GEAR



Since the distance l is in both cases nearly the same, the difference in both times is essentially determined by the stress difference in both cases. In the case of the off-center landing represented by Figure 52, the times which the transverse waves need to reach the deck sheaves on the starboard and port sides are considerably different and longer because of the large span in this case. In Figures 51 and 54, point (4) denotes the stress due to impact at the sheave, including the reflection from the hook as computed from the theory.

The kink produced by the impact of the cable at the sheave S (see Figure 56) propagates toward the hook at P and upon arrival at this point is reflected (see Figure 58) in a manner which can be computed from the considerations of Section IV 10. In general, however, the increase in tension due to this reflection is not considerable, but the sudden change of the angle which the cable forms at the hook results in a considerable change of the load exerted by the cable on the hook of the airplane. The kink due to reflection at the hook P propagates toward the sheave S producing here another impact and so on.

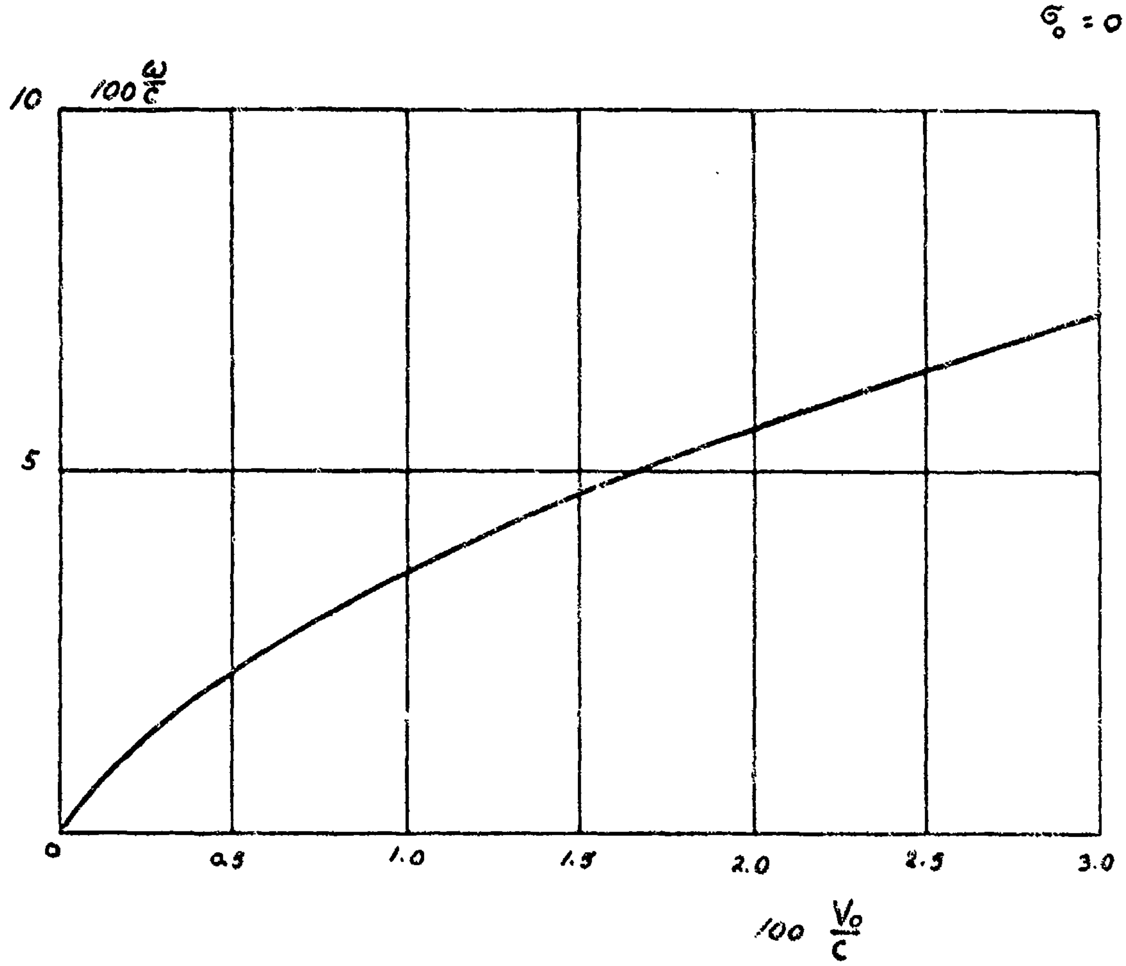


FIGURE 58
KINK VELOCITY FOR TRANSVERSE IMPACT

5. Average Cable Tension and Equations of Motion of a Conventional Arresting Gear

If we connect the sheave S with the hook P by a straight line (dashed line in Figure 59) this line moves as point P is moving. It coincides with the position of the cable always when the kink has reached the sheave or the hook as indicated, for instance, by the special points P_1 and P_2. The figure shows that the cable section between S and P performs a transverse vibration about the dashed line SP.

We now neglect this vibration, replacing the actual motion of the cable between points S and P by a straight cable SP, rotating around point S in any moment. We further neglect any longitudinal vibration of the cable which might take place between the hook point P and the anchor. In other words, we assume that the tension of the cable at any time t is a constant along the total cable changing only with the time t. This tension $\bar{T}$ we denote as the average tension in the arresting cable.

Under these simplified assumptions, we determine the equations of the motion and cable tension of a conventional arresting gear as represented schematically by Figure 60. The figure shows the geometry at any time t. The cable length between hook point and anchor in battery position is denoted by L, the half span by l, the airplane mass by m, the mass of the moving crosshead, including sheaves and ram by M and the number of cables pulling at the crosshead at one engine side (the reeving ratio) by $2n$. It is assumed that the engine is reeved symmetrically and that the landing is on center. At the crosshead other $2n$ cables are pulling from the other side. They are not shown in the figure. The force acting against the crosshead motion due to the cylinder pressure is F.

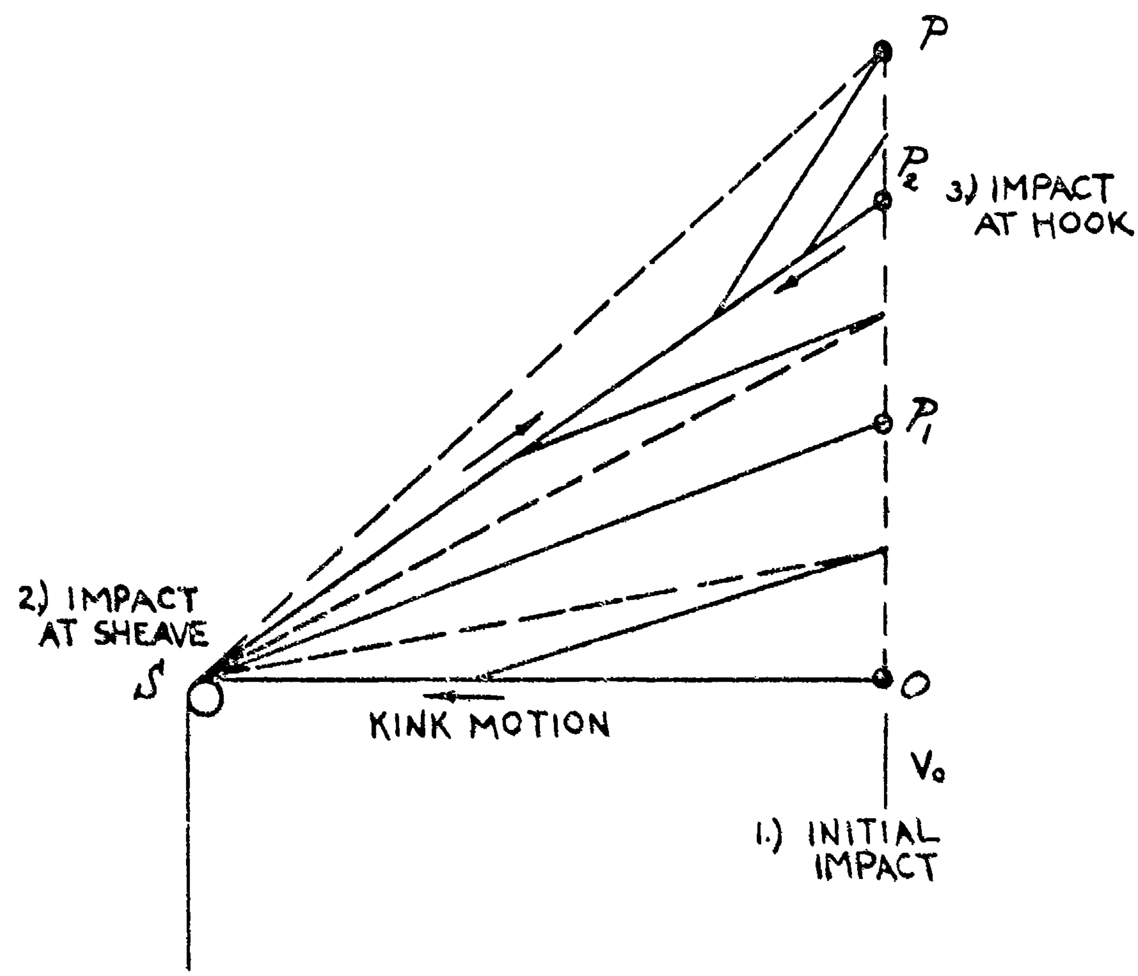

P
P_2
3.) IMPACT
AT HOOK
P_1
2.) IMPACT
AT SHEAVE
S
O
KINK MOTION
V_o
1.) INITIAL
IMPACT

FIGURE 59

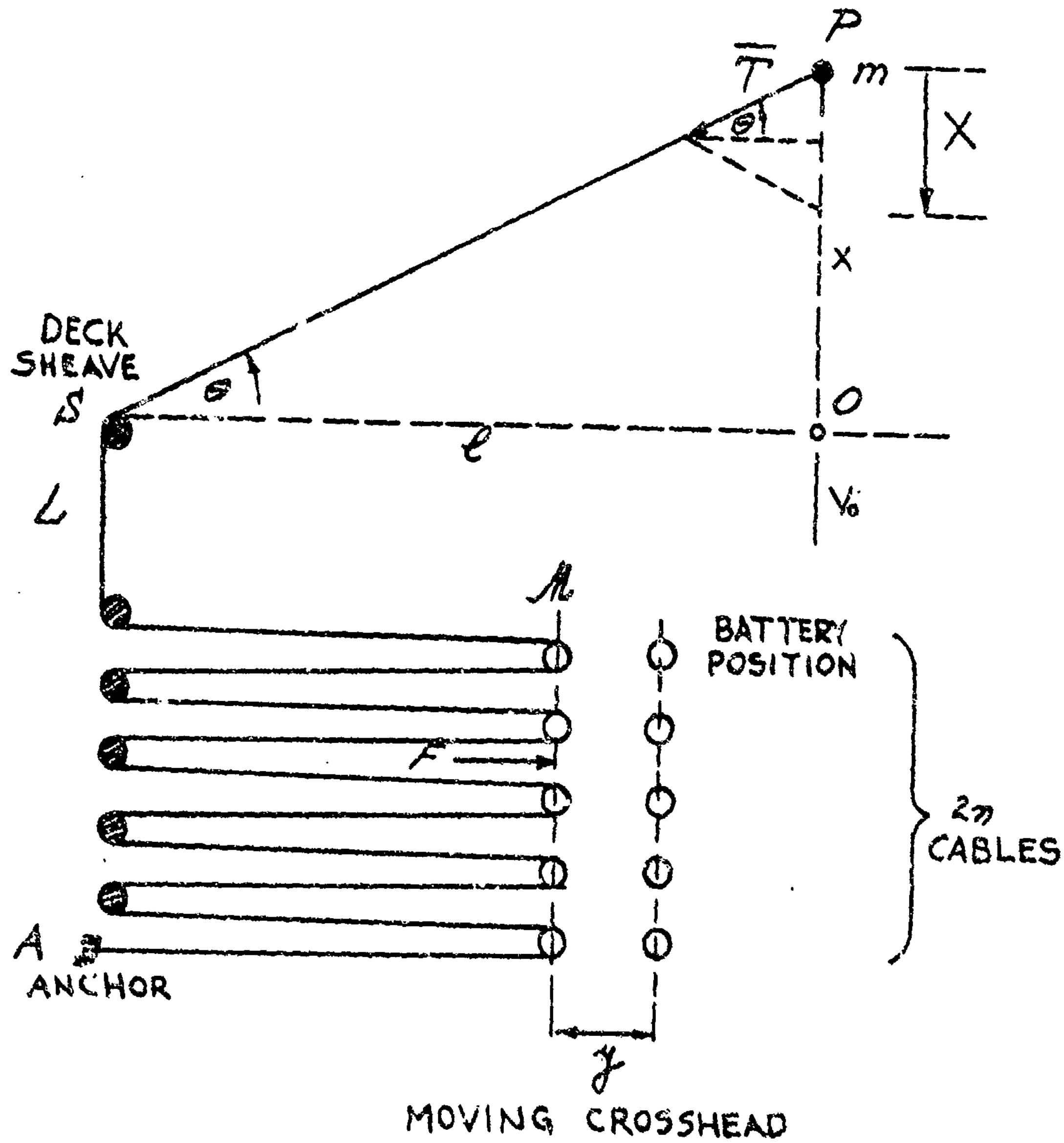

P
$\overline{T}$
m
θ
X
x
DECK
SHEAVE
S
θ
O
ℓ
V_0
L
M
BATTERY
POSITION
F
2n
CABLES
A
ANCHOR
y
MOVING CROSSHEAD

FIGURE 60

Acting against the airplane motion is the force X due to the tension $\bar{T}$. If Θ is the angle between the cable at time t and its position SO at time zero we have

$$\frac{X}{2\bar{T}} = \sin\Theta = \frac{tg\,\Theta}{\sqrt{1 + tg^2\Theta}}$$

and since

$$tg\,\Theta = \frac{x}{\ell}$$

x being the runout of the airplane at time t we find

$$X = 2\bar{T}\frac{x}{\sqrt{x^2 + \ell^2}}. \tag{205}$$

According to Newton's law, the equation of motion of the airplane

$$m\ddot{x} = -X.$$

Replacing X by the expression (205) we get

$$\boxed{\bar{T} = -\frac{m}{2}\frac{\ddot{x}}{x}\sqrt{x^2+\ell^2}} \tag{206}$$

where $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt}$, v being the velocity of the airplane.

During the time t the crosshead will have made the stroke y. Thus, the elongation of the cable due to the tension $\bar{T}$

$$\delta = \sqrt{x^2+\ell^2} - \ell - 2ny$$

and according to Hooke's law

$$\boxed{\bar{T} - T_0 = \frac{qE}{L}\left(\sqrt{x^2+\ell^2} - \ell - 2ny\right)} \tag{207}$$

where T_0 is the initial tension, q the cross section of the cable and E its elasticity modulus.

The crosshead moves under the action of the tension $\bar{T}$ in the $4n$ cable segments pulling at it and the action of the force F due to the cylinder pressure. Thus, the equation of motion of the crosshead is

$$\boxed{M\ddot{y} = 4n\bar{T} - F} \tag{208}$$

according to Newton's law.

If F is a given function of time t, the three equations (206), (207), and (208) determine the three unknown functions $x = x(t)$, $y = y(t)$ and $\bar{\bar{T}} = \bar{\bar{T}}(t)$. Replacing $\bar{\bar{T}}$ in equation (207) by the expression (206) from equation (206) we obtain together with (208) two non-linear second order differential equations for the unknown strokes x and y as functions of time. No exact general solution of this system for which the initial conditions are

$$x = 0 \,, \quad \dot{x} = V_0 \,, \quad y = 0 \,, \quad \dot{y} = 0 \tag{209}$$

at time $t = 0$ is known. However, computer solutions will be available in the near future. Approximate solutions under special conditions can be obtained by existing methods. For instance, in the case of low reeving ratios, such solutions have been discussed in detail by Robert S. Ayre, using the phase-plane-delta method *.

For the design of an arresting gear considerably more useful is the solution of the problem to determine F for an average tension $\bar{\bar{T}}$ prescribed as a function of the runout x. This problem can be solved exactly in the following manner. If $\bar{\bar{T}}(x)$ is given equation (206) can be written in the form

$$\bar{\bar{T}}(x) = -\frac{m}{2}\,\frac{\ddot{x}\,\dot{x}}{\dot{x}\,x}\,\sqrt{x^2 + \ell^2}$$

or

$$\frac{d(\dot{x}^2)}{d(x^2)} = -\frac{2}{m}\,\frac{\bar{\bar{T}}(x)}{\sqrt{x^2 + \ell^2}}$$

from which follows

$$V^2 = V_0^2 - \frac{4}{m}\int_0^x \frac{x\,\bar{\bar{T}}(x)}{\sqrt{x^2 + \ell^2}}\,dx \tag{210}$$

where $V = \dot{x}$ is the velocity of the airplane the initial velocity at the time $t = 0$ being V_0. The time t belonging to the runout is then given by

* Reference 15

$$t = \int_0^x \frac{dx}{\sqrt{V_0^2 - \frac{4}{m}\int_0^x \frac{x\,\bar{T}(x)}{\sqrt{x^2+\ell^2}}\,dx}} . \qquad (211)$$

The stroke y of the crosshead follows from equations (207) which yields

$$y = \frac{1}{2n}\left[\sqrt{x^2+\ell^2} - \ell - \frac{L}{qE}\left(\bar{T}(x) - T_0\right)\right]. \qquad (212)$$

The velocity of the crosshead

$$\dot{y} = \frac{1}{2n}\left[\frac{x}{\sqrt{x^2+\ell^2}} - \frac{L}{qE}\,\bar{T}'(x)\right] V \qquad (213)$$

where

$$\bar{T}' = \frac{d\bar{T}}{dx}.$$

Differentiation with respect to t yields the acceleration of the crosshead

$$\ddot{y} = \frac{1}{2n}\left[\left(\frac{\ell^2}{\sqrt{\ell^2+x^2}^{\,3}} - \frac{L}{qE}\,\bar{T}''(x)\right)V^2 + \left(\frac{x}{\sqrt{x^2+\ell^2}} - \frac{L}{qE}\,\bar{T}'(x)\right)\dot{V}\right]. \qquad (214)$$

V^2 is given by (210). $\dot{V} = \ddot{x}$ follows from (206):

$$\dot{V} = -\frac{2}{m}\,\frac{x\,\bar{T}(x)}{\sqrt{x^2+\ell^2}} . \qquad (215)$$

Thus the unknown F follows from equation (208) in the form

$$F(x) = 4n\,\bar{T}(x) - \frac{M}{2n}\left[\left(\frac{\ell^2}{\sqrt{x^2+\ell^2}^{\,3}} - \frac{L}{qE}\,\bar{T}''(x)\right)\left(V_0^2 - \frac{4}{m}\int_0^x \frac{x\,\bar{T}(x)}{\sqrt{x^2+\ell^2}}\,dx\right)\right.$$
$$\left. - \frac{2}{m}\left(\frac{x}{\sqrt{x^2+\ell^2}} - \frac{L}{qE}\,\bar{T}'(x)\right)\frac{x\,\bar{T}(x)}{\sqrt{x^2+\ell^2}}\right]. \qquad (216)$$

<u>This formula determines the force F as function of the runout x which is required in order to obtain the prescribed average tension $\bar{T}(x)$.</u>

At the time $t = 0$ the pressure force $F(0)$ and the force $4n\,T_0$ due to cable tension are supposed to be in equilibrium at the crosshead so that $\ddot{y} = 0$ for $t = 0$. Since there is at the time $t = 0$ no deceleration of the airplane under the simplifying assumptions upon which this consideration is based formula (214) yields

$$\bar{\bar{T}}''(0) = \frac{qE}{\ell L} \tag{217}$$

as additional condition to be satisfied by the function $\bar{\bar{T}}(x)$.

For conventional arresting gears with high reeving ratio, for instance the Mark 7 arresting gear with $2n = 18$, the right side of formula (216) is not very different from $4n\,\bar{\bar{T}}(x)$ so that approximately

$$F(x) = 4n\,\bar{\bar{T}}(x). \tag{218}$$

In Figure 53 which represents a Mark 7 measurement, the curve $\frac{F(t) - F(0)}{4n}$ versus time t has been plotted. The values of $\frac{F(t) - F(0)}{4n}$ are computed from

$$\frac{F(t) - F(0)}{4n} = \frac{314\,(p - p_0)}{36}$$

using the measured values of the cylinder pressure curve as prescribed values. According to the preceding discussion, this curve approximately represents the average tension values $\bar{\bar{T}} - T_0$. On the other hand we can consider the average tension $\bar{\bar{T}}$ approximately as the mean value of the anchor tension and tension measured in the deck link. At the time corresponding to point 2 the anchor tension is about zero. Thus, the tension in the deck link must be approximately twice the average tension value. In this way, point 5 has been computed.

6. Tension Vibrations in the Arresting Cable

The tension in the arresting cable at any time t is far from being constant along the cable as a comparison of the tension at the anchor with the tension at the deck link shows as indicated in Figure 54 for instance. This tension difference is caused by the fact that a disturbance propagates with the finite velocity c . Disturbances are produced by the initial impact when the hook engages the cable, the impacts of the cable at the deck sheave, the impacts of the cable at the hook and especially the longitudinal impact occurring at the anchor in the moment when the slack is picked up completely.

The initial impact produces after a short time slack (zero tension) in the cable at the anchor and in the case of a conventional arresting gear which is picked up completely at a time t^* determined before. The longitudinal motion of the cable is in this moment suddenly stopped by the anchor resulting in a sudden increase in tension. From this moment on, any disturbance propagates with the speed c unless new slack is produced which is normally not the case and shall be excluded for the following discussion. Then the anchor impact tension arrives at the deck link at the time $t^* + \frac{L-l}{c}$ and is reflected at the hook at the time $t^* + \frac{L}{c}$ (see Figure 54). The reflected wave arrives at the anchor at the time $t^* + 2\frac{L}{c}$ and so on.

The first impact at the deck sheave occurs at the time $t_1 = \frac{l}{\omega_1}$ where ω_1 is the average transverse wave velocity during the travel from the center span to the sheave. If the slack has been picked up before this impact, the resulting tension disturbance travels with the velocity c too. If we neglect for this tension propagation the distance between sheave and hook both disturbances have a phase difference of $t^* - t_1$.

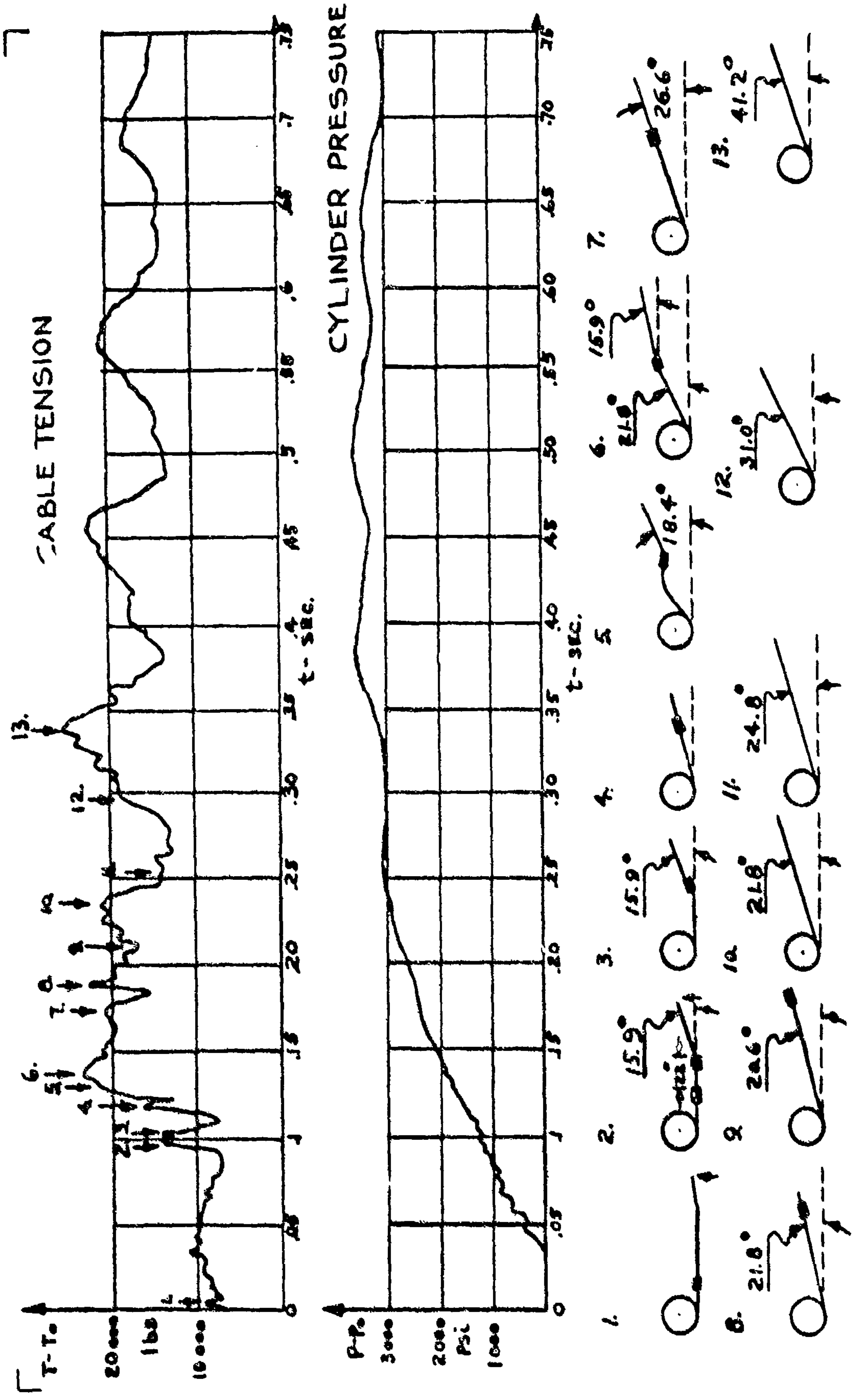

CABLE TENSION
CYLINDER PRESSURE
T-T₀
20000
lbs
10000
P-P₀
3000
2000
psi
1000
t - SEC.
0
.05
.1
.15
.20
.25
.30
.35
.40
.45
.50
.55
.60
.65
.70
.75
15.9°
18.4°
21.8°
15.9°
26.6°
26.6°
21.8°
24.8°
31.0°
41.2°

FIGURE 61

The next disturbance occurs when the transverse wave which is reflected at the sheave arrives at the hook. The required time is $t_2 = \frac{l_1}{\omega_2}$ where l_1 denotes distance between sheave and hook at time t_1 and ω_2 the average transverse wave velocity during the travel from the sheave to the hook and so on. The disturbance travels with the velocity c having a phase difference $t_2 - t_1$ against the preceding disturbance and so on. With increasing time, the distance between sheave and hook will not be negligible anymore for the computation of the propagation of the disturbance. However, later the disturbances will be damped out anyhow and will have no interest anymore the vibrations assuming more and more at random character.

At conventional arresting gears, another disturbance is produced by the deck link connecting the deck pendant with the purchase cable. The mass of such link is considerably larger than the mass of a cable segment of the same length. According to the investigations of III 4(d), the longitudinal stress propagation is disturbed by such link during a very short time only. However, there is a considerable disturbance resulting from a transverse wave hitting the deck link. The magnitude of the impact stress obtained in this case can be computed from the oblique impact formula. Figure (61)* shows a deck link tension measurement from a Mark 5 arresting gear in comparison with the deck link and cable motion obtained from high speed movie pictures. The considerable tension peak (points 2 and 3) due to the impact of the cable at the link should be noted.

The phase differences between the stress waves discussed before are important for the design of conventional arresting gears. If, for instance,

$$t_1^* - t_1 = 0$$

the initial stress wave returning from the anchor in form of an anchor impact wave coincides with the stress increase due to cable impact at the deck sheave at the time t_1, the result is a superposition of high tension peaks in

*Compare reference 17

the deck pendant which reduces the applicable engaging speed of the airplane. In the first Mark 5 arresting gear design, this has been exactly the case. For a fixed engaging speed such coincidence can be eliminated by changing the cable length. However, since the time t_1 depends on the stress in the cable and, therefore, on the engaging speed while (for constant runout arresting gears) t^* is approximately constant an optimum cable length can be obtained as a compromise for a certain speed range only.

Finally a different type of stress vibration has to be mentioned. It is that of the average tension $\bar{T}$ due to the mass of the crosshead. If the mass M of the crosshead is large, its response to an increasing or decreasing tension $\bar{T}$ will be slow. Thus the tension will increase to a higher value before it starts moving and will drop to a lower value before it stops moving compared with the case of a small M. This shows that the cable and the mass of the crosshead form a vibrating system. In order to determine this vibration and especially its frequency, we assume that $x = x(t)$ is known and replace $\bar{T}$ in equation (208) from equation (207) which yields

$$\ddot{y} + \frac{8n^2 q E}{LM} y = \varphi(t) \tag{219}$$

where

$$\varphi(t) = \frac{4n}{M}\left[T_0 + \frac{qE}{L}\left(\sqrt{x^2+l^2} - l\right)\right] - \frac{F(t)}{M}. \tag{220}$$

This is the differential equation of a forced vibration. Its solution with the initial conditions

$$y = 0, \qquad \dot{y} = 0 \qquad \text{for} \qquad t = 0$$

is

$$y = \frac{1}{\nu}\left[\sin\nu t \int_0^t \varphi(t)\cos\nu t\,dt - \cos\nu t \int_0^t \varphi(t)\sin\nu t\,dt\right] \tag{221}$$

where

$$\nu = \sqrt{\frac{8n^2 q E}{LM}} . \qquad (222)$$

Replacing now y in equation (207) by this solution we obtain

$$\bar{T} = T_0 + \frac{qE}{L}\left[\sqrt{x^2+l^2} - l - \frac{2n}{\nu}\left(\sin\nu t\int_0^t \varphi(t)\cos\nu t\,dt - \cos\nu t\int_0^t \varphi(t)\sin\nu t\,dt\right)\right] . \qquad (223)$$

For the time immediately after the engagement we have approximately

$$x = v_0 t , \qquad F = 4nT_0 + kt^2 \qquad (224)$$

where k is a constant. Thus for small t values

$$\varphi(t) = \kappa(\nu t)^2 \qquad (225)$$

where

$$\kappa = \frac{v_0^2}{4ln} - \frac{1}{\nu^2}\frac{k}{M} . \qquad (226)$$

With this function φ the integrals in (223) can be evaluated analytically and yield the result

$$\bar{T} = T_0 + \frac{qE}{L}\left[\sqrt{x^2+l^2} - l + \frac{2n\kappa}{\nu^2}\left(2 - 2\cos\nu t - (\nu t)^2\right)\right] . \qquad (227)$$

This result shows that the average tension $\bar{T}$ contains a vibration $\cos\nu t$ where ν is determined by formula (222). In the case of the Mark 7 arresting gear where

$$2n = 18 , \quad q = 0.00519 \ ft^2 , \quad E = 18.3 \cdot 10^8 \ lbs/ft^2 ,$$

$$L = 786 \ ft , \quad M = 447 \ lbs \cdot sec^2/ft$$

we obtain from this formula

$$\nu = 132.4$$

which means a rather high frequency.

7. Multicylinder Arresting Engine

In order to avoid tension peaks in limited areas of the arresting cable which will reduce the performance of the arresting gear, we have to try to distribute the amount of energy transmitted to the cable in form of strain energy, for instance by an impact, immediately over the total cable. Several methods to do this have been proposed by the author of the present monography and shall be discussed briefly in the following sections. The first method consists in using a large number of small cylinders instead of a large cylinder as indicated in the schematical figure 62. The piston of each cylinder carries one crosshead sheave. All pistons are able to move independently. All cylinders are close together and connected by pipes so that the fluid can move freely from one cylinder into another. The cable is reeved like a conventional arresting engine.

Any tension increase in the deck pendant ~~propagates~~ to the sheave attached to the first piston and pushes this into the first cylinder. The rising pressure in the liquid propagates into the other cylinders pushing the pistons out and produces tension in the cable. Though the velocity of the pressure propagation in a liquid is only about one-half of that of the stress propagation in the cable, the energy transmitted to the cable in form of strain energy is distributed nearly at the same time over the total cable because the distance between the pistons in the liquid are due to the compact arrangement of the cylinders by far shorter than the distance along the cable. The pressure is controlled by a valve in any conventional way. Any disturbance of the tension at any place in the system is distributed by this device in a very short time over the total cable so that superpositions of tension peaks never can occur. Since the total piston area can be made large without increasing the mass of the single crossheads and pistons too much, this arresting engine can be operated at considerably lower pressures compared with conventional engines of the same performance.

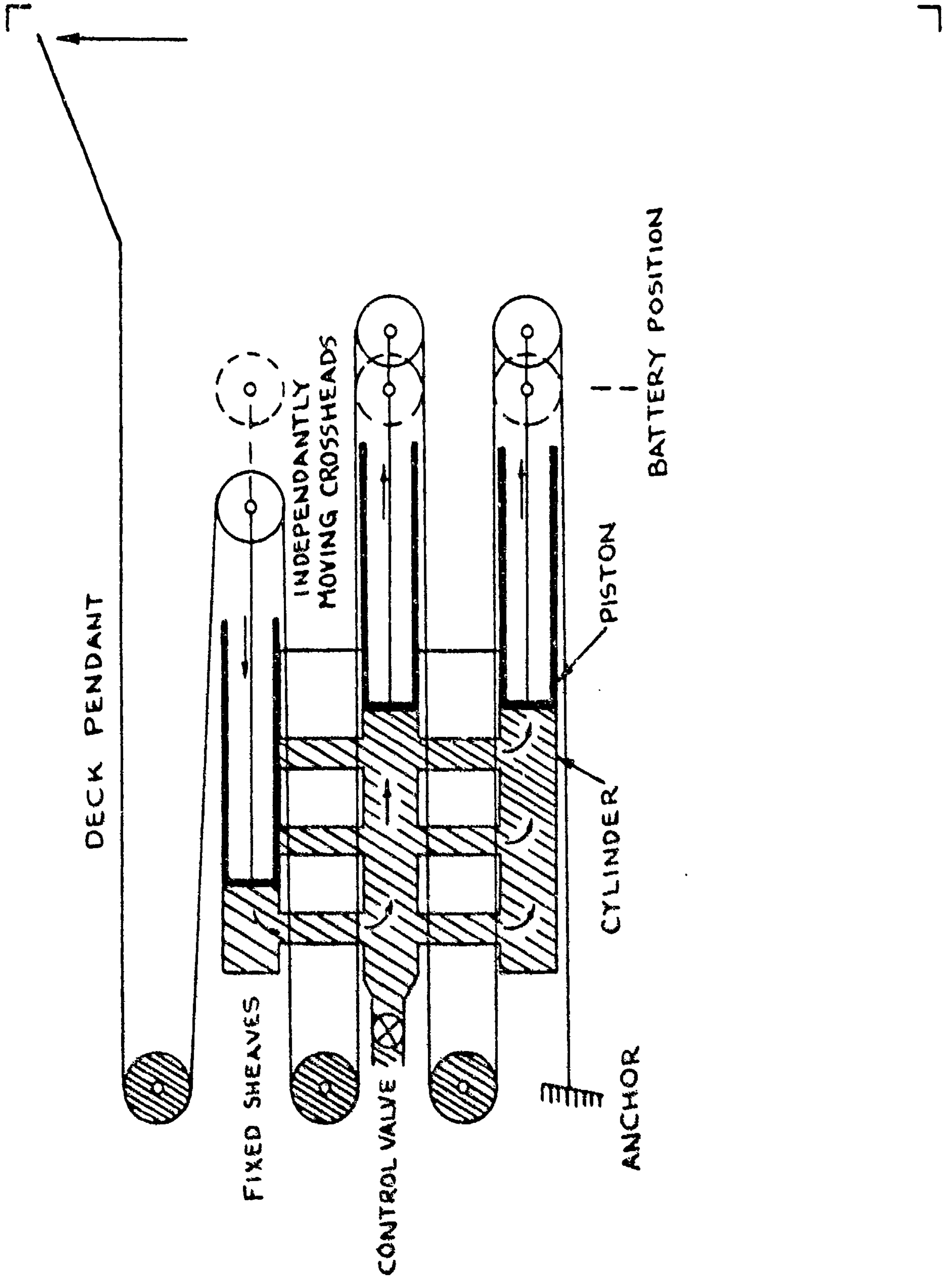

DECK PENDANT
INDEPENDANTLY
MOVING CROSSHEADS
BATTERY POSITION
PISTON
CYLINDER
FIXED SHEAVES
CONTROL VALVE
ANCHOR

FIGURE 62

Because the tension in the cable will not deviate considerably from the average cable tension as defined in section V5, the required pressure can be computed directly from the equations of this section without particular consideration of disturbances due to impacts. No slack will ever form in this system.

8. Slackless Reeving Systems and Moving Sheaves

A second method for distributing an energy amount in form of strain energy simultaneously over the total cable in a more or less perfect way consists in the use of slackless reeving systems. For instance, a reeving system as shown in Figure 63 consists of the conventional moving crosshead and a conventional set of fixed sheaves. In between pairs of sheaves are arranged which can move along short tracks. The cable is reeved as indicated in the figure. A stress produced in the deck pendant propagates now in a very short time along the cable segment which runs in turn over a fixed sheave and one of the movable pairs because this cable segment is very short in comparison with the total cable length. The stress in this cable segment is transmitted also in a very short time to the cable segments between the sheaves of the moving crosshead and the other sheaves of the moving pairs. Similarly, any disturbance at any place in the system is distributed in a short time over the total cable. The tracksfor the moving pairs have to be short because they have to move only about a distance which is sufficient to equalize the tensions in the cable segments.

Examples for other slackless reeving systems are represented in Figure 64 where fixed sheaves are shaded. Here, however, tracks of considerable length for the moving pairs are required. The longest must have the length of about one-half of the total runout.

The slackless reeving systems discussed before are characterized by the use of moving sheaves. If the installation of such system at an existing engine is impossible, single moving sheaves connected with a separate damper or shock absorber might be feasible and useful. For instance, a system as shown in Figure 65 can be used to eliminate the adverse effect of the cable impact at the deck sheave which usually produces the highest tension peak. S_1 is the deck sheave. Between S_1 and another fixed sheave S_2

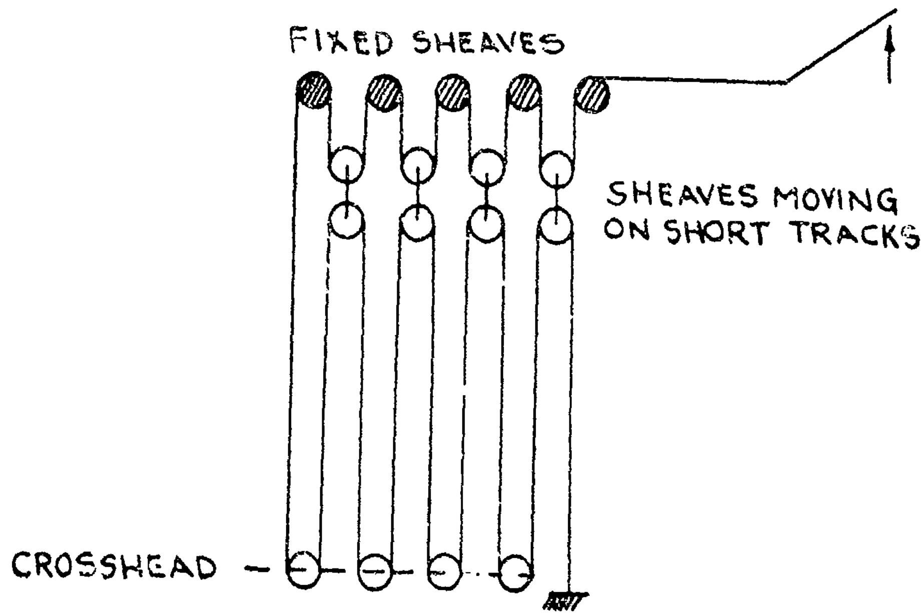


FIGURE 63

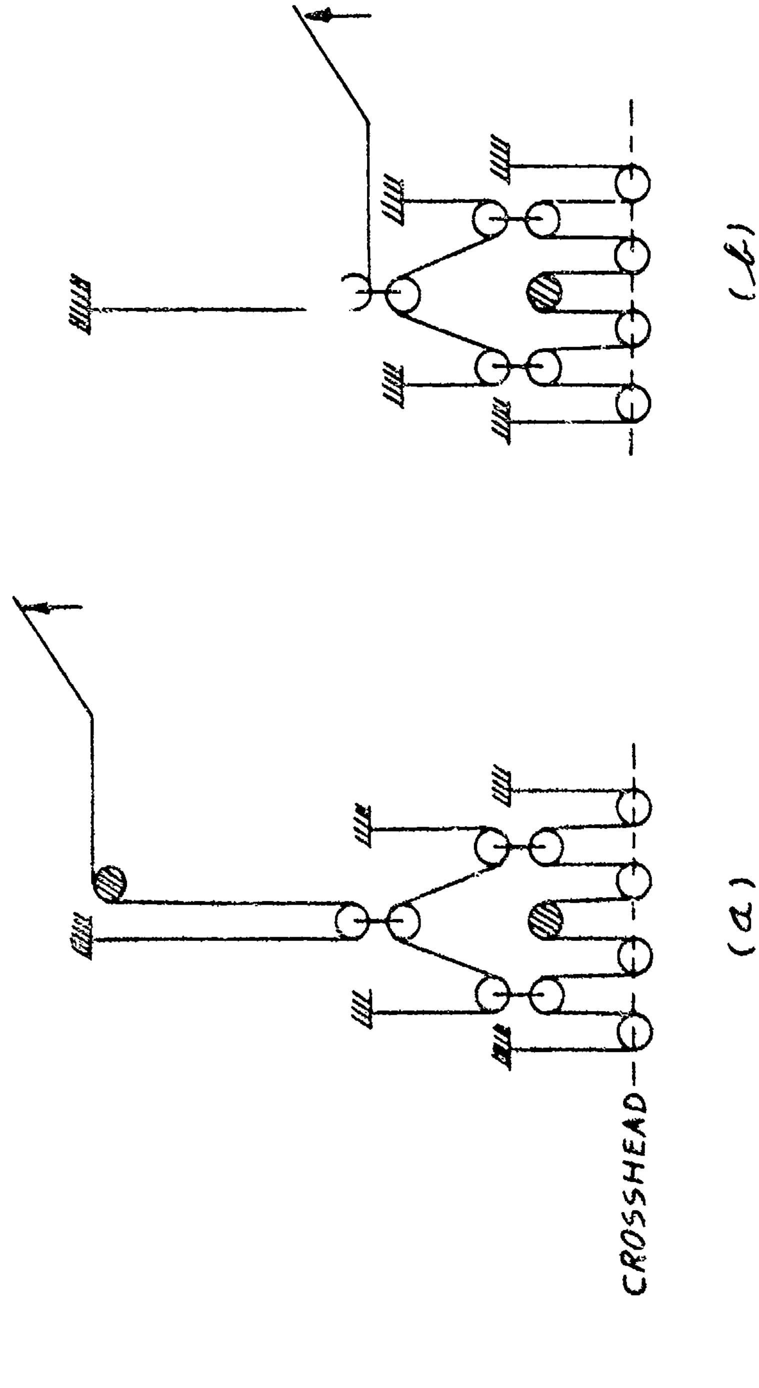


FIGURE 64

a movable sheave S connected with the piston of a shock absorber is installed. By controlling the pressure p in the shock absorber automatically the incoming initial impact tension or the tension increase due to impact of the cable at the deck sheave can be reduced to a desired degree. Analytical and experimental investigation of such and similar systems are in preparation.

We denote by T_1 the tension in the cable due to the initial impact, by λ_0 the length of the cable segment $S_1 S_0 = S_2 S_0$ where S_0 is the initial position of the movable sheave S and by α the angle between $S_1 S$ and the axis of the piston. m is the mass of the moving sheave including ram and piston. We approximately determine here the required force F as function of the time t in order to produce a given tension T in the cable due to the motion of the sheave. We can assume that the tension due to the motion at the time t is distributed equally along the cable between the hook point and the point in the distance ct from the hook point. Then

$$T_1 - T = 2\,\frac{\lambda - \lambda_0}{ct}\,q E \tag{228}$$

because $2(\lambda - \lambda_0)$ is the cable length about which the cable length ct has been shortened due to the sheave motion. From the geometry of Figure 65 follows further

$$\frac{\lambda}{\lambda_0} = \frac{\sin\alpha_0}{\sin\alpha} \tag{229}$$

and

$$\frac{y}{\lambda_0} = \cos\alpha_0 - \sin\alpha_0 \operatorname{ctg}\alpha . \tag{230}$$

The equation of motion of the sheave is

$$m\ddot{y} = 2T\cos\alpha - F(t) \tag{231}$$

If now T is a prescribed function of t equation (228) determines $\lambda = \lambda(t)$ From equation (229) follows $\alpha = \alpha(t)$. Thus, from equation (230) $y = y(t)$ is known and equation (231) determines subsequently the required force

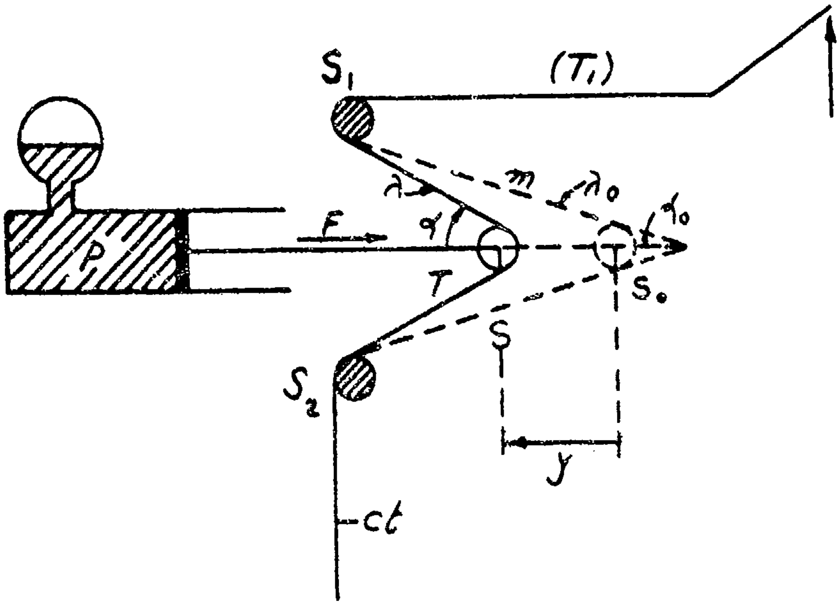


FIGURE 65

$$F = F(t)$$

from which the required cylinder pressure p as function of t or as function of the stroke y can be computed. This computation is valid only for a time during which no reflected stress wave returned to the moving sheave. After this time the computation can be continued similarly under new initial conditions.

By a corresponding approximate method the required force F can be determined which has to act against a movable deck sheave in order to obtain a prescribed cable tension (see Figure 66). Here we assume that the deck sheave with the mass m including all movable parts starts its motion perpendicular to the deck pendant along a track under the influence of the tension T_1 produced by the initial impact the force F acting in opposite direction.

Since the mass m starts its motion from zero velocity under the influence of T_1 we can assume that the angle β of the deck pendant after a short time t is negligibly small. Then m is moving under the force $T - F$. Due to the motion of the sheave, about the amount y the cable is shortened about the same amount. Thus

$$T_1 - T = \frac{y}{ct} qE \tag{232}$$

where again is assumed that the tension T is spread uniformly over the length ct. The equation of motion of the sheave is

$$m\ddot{y} = T - F(t). \tag{233}$$

If now T is a prescribed function of the time t equation (232) determines y as function of t. Thus $F(t)$ can be computed from equation (233). Eliminating y from the equations (232) and (233) we obtain for the required force

$$F(t) = T + \frac{mc}{qE}(2\dot{T} + t\ddot{T}). \tag{234}$$

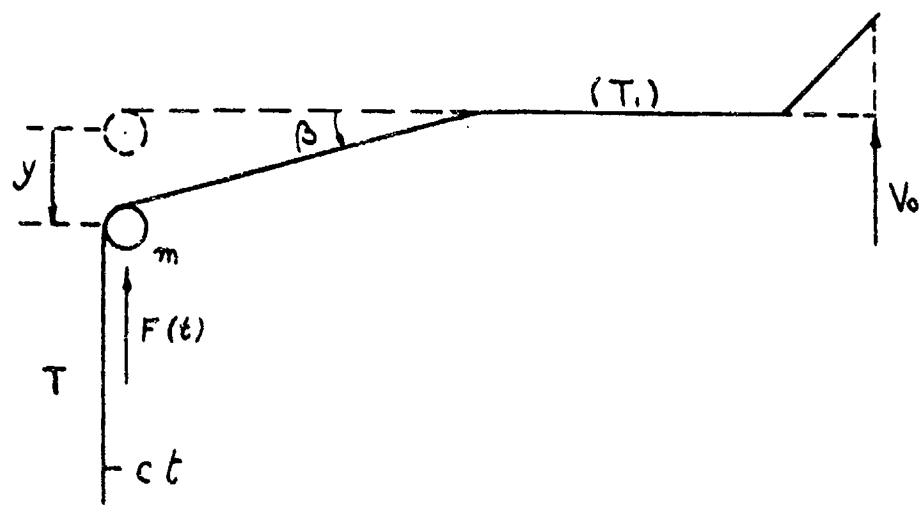


FIGURE 66

This result is valid only for a short time after the start of the motion of the sheave according to the simplifying assumtions made before and at most until the transverse wave hits the moving sheave.

Of particular interest is the case

$$T = T_1 e^{-\lambda t} \tag{235}$$

where T is exponentially decreasing with the time t.

In this case formula (234) yields

$$F = \left(1 - 2\lambda \frac{mc}{qE} + \lambda^2 \frac{mc}{qE}\right) T_1 e^{-\lambda t}. \tag{236}$$

For practical applications the controlling force F will have to be zero at the moment $t = 0$ where the tension T_1 reaches the sheave because otherwise it would have to jump exactly in this moment to the required value which seems difficult to realize. This condition can be satisfied in the present case by chosing

$$\lambda = \frac{1}{2} \frac{qE}{mc}. \tag{237}$$

The force F is then determined by

$$F = \frac{\lambda}{2} t T_1 e^{-\lambda t}. \tag{238}$$

This result is represented for a 1-3/8" cable and sheave weights of 1000, 2000 and 4000 pounds in Figures 67 and 68. The first of these figures shows the decreasing tensions in the three cases. The second represents the corresponding control forces F.

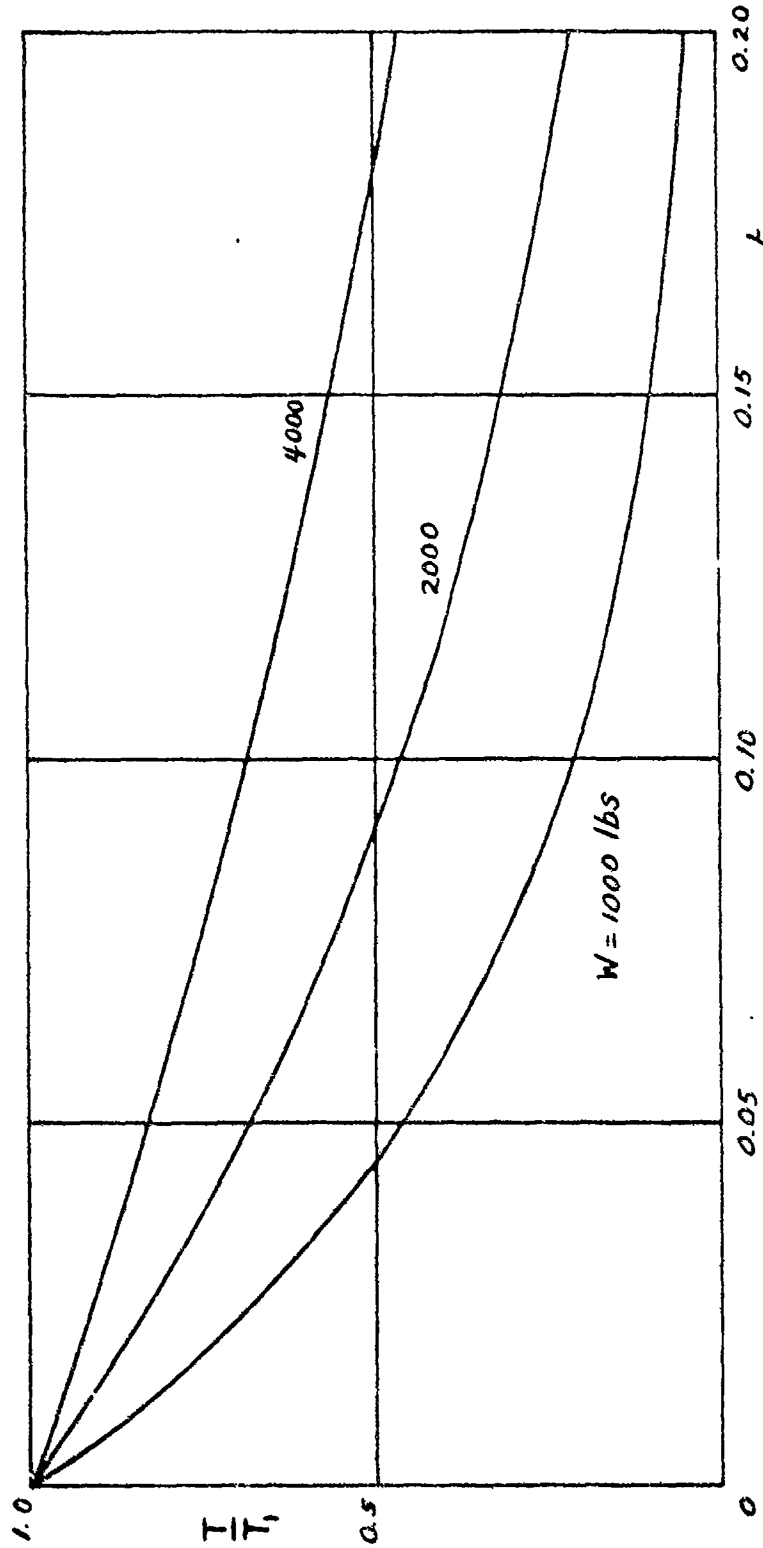


FIGURE 67

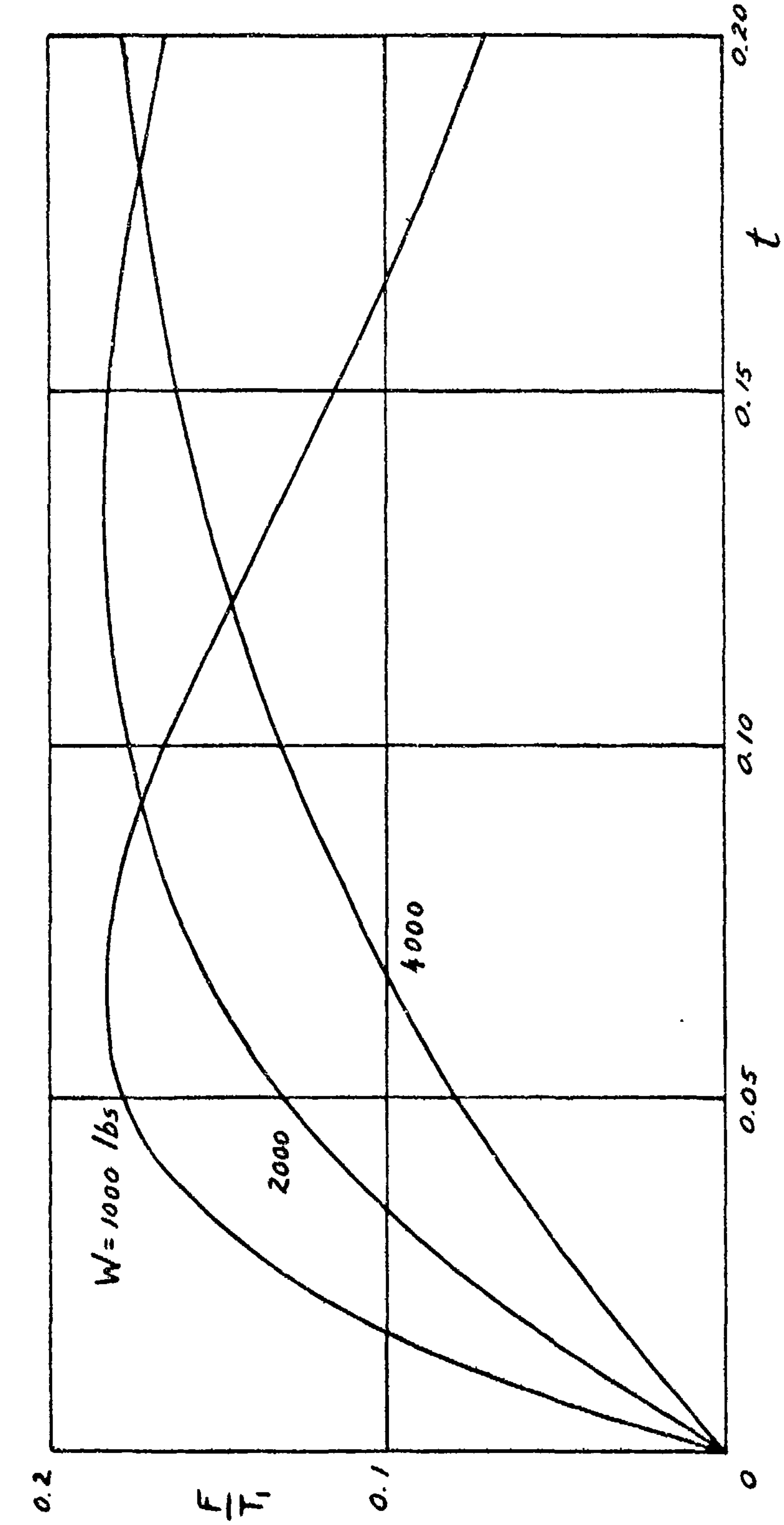


FIGURE 68

9. The Water Squeezer

Closely related to the last problem of the preceding section is the starting problem of a simple type of hydraulic arresting engine presently under development at All American Engineering Company, Wilmington, Delaware, called the "water squeezer." It consists basically of a long closed tube of variable inside cross section or any equivalent device through which a floating piston is pulled directly by the arresting cable (see the schematic figure 69). The cable seals the tube which is filled with water. The orifice between piston and wall of the tube provides the controlling force F required for a desired cable tension.

The tension T_1 due to the initial impact at the deck pendant propagates along the cable and is reflected completely at the mass m of the piston so that the piston starts to move due to the tension $2T_1$. If x is the stroke of the piston after a short time t the cable segment between the piston and the point in the distance ct is shortened about the amount x. Thus the tension T in this cable segment is given by

$$2T_1 - T = qE\frac{x}{ct}. \tag{239}$$

The equation of motion of the piston is

$$m\ddot{x} = T - F(t) \tag{240}$$

as long as no new tension wave arrives at the piston. If we eliminate x from these two equations we obtain

$$F(t) = T + \frac{mc}{qE}(2\dot{T} + t\ddot{T})$$

which is identical with equation (234) of the preceding section.

Thus in particular the result represented by the Figures 67 and 68 can be applied directly to the water squeezer with the difference only that T_1 has to be replaced by $2T_1$. A satisfactory performance, however, can be

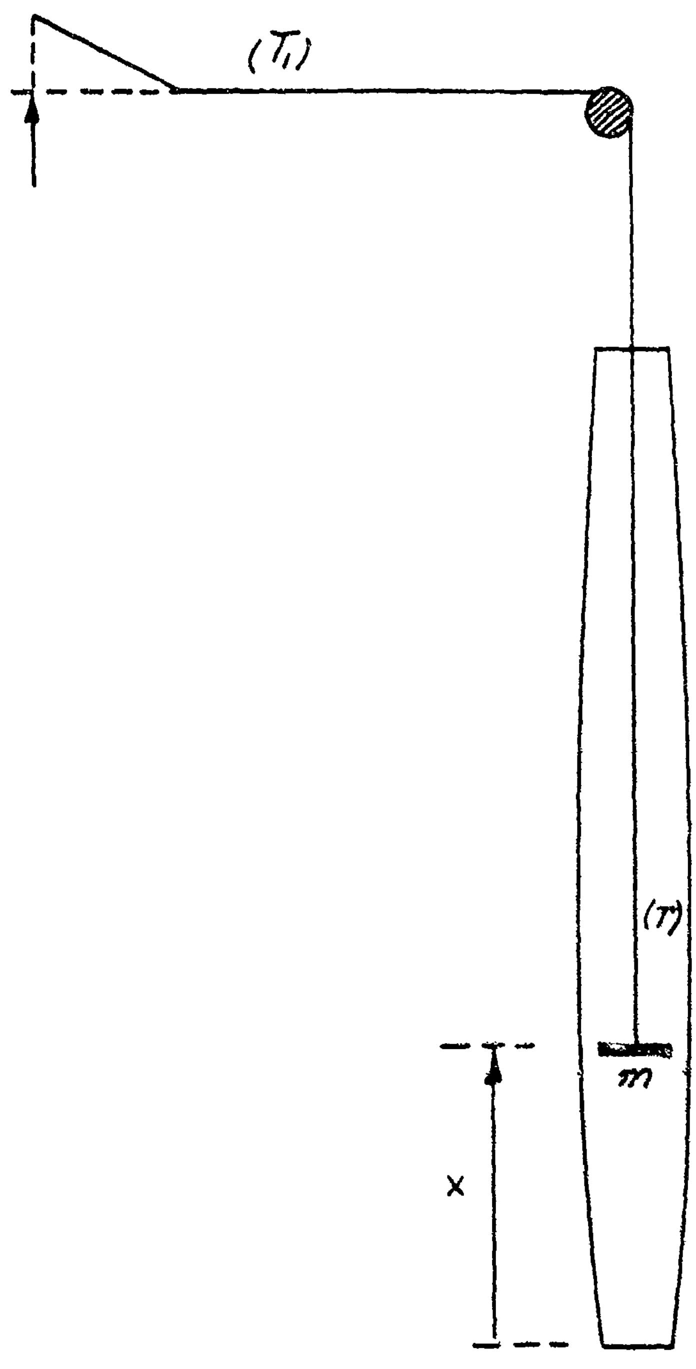


FIGURE 69

expected only if the tension T' which is lower than $2T_1$ arrives at the deck pendant before the transverse impact wave reaches the deck sheave. Unless the deck span has a considerable length too, if the water squeezer is long, the tension $2T_1$ will be superposed by the impact tension at the deck sheave. At a conventional deck span a long water squeezer will give satisfactory results only if it is coupled with a moving deck sheave or a moving sheave device as represented by Figure 65.

10. Three-Dimensional Cable Motion. Double Wire Engagements

The investigations of this monography have been restricted to two-dimensional cable motions up to this point. However, the method of analyzing such motion by a sequence of oblique impacts is applicable also to three-dimensional problems. This will be explained in the present section at two problems arising with the airplane arrestment.

The tail hook of the airplane to be arrested has in the moment of cable engagement a downward position forming with the deck an angle which can be as large as 90° depending on the installation of the hook at the airplane and the position of the landing airplane against the deck (see Figure 70). For simplicity let us assume that the airplane moves parallel to the deck from the moment of engagement on. If we neglect the mass of the hook, its direction subsequently will be that of the force which the cable exerts on it. The motion of the hook point is, therefore, that described by a tractrix.

We denote the length of the hook shank by l, the distance of its pivot point from the deck by h and its variable angle with the deck by α. In the moment of engagement at the time $t = 0$ the value of α is denoted by α_0 which is a given value. The x - axis of a rectangular coordinate system shows into the direction of the moving airplane, the y axis being perpendicular to the deck. The origin O is the engaging point of the cable. From Figure 70 follows then

$$x = v_0 t + l \cos\alpha_0 - l \cos\alpha$$

$$y = h - l \sin\alpha$$

$$\frac{y'}{x'} = tg\,\alpha$$

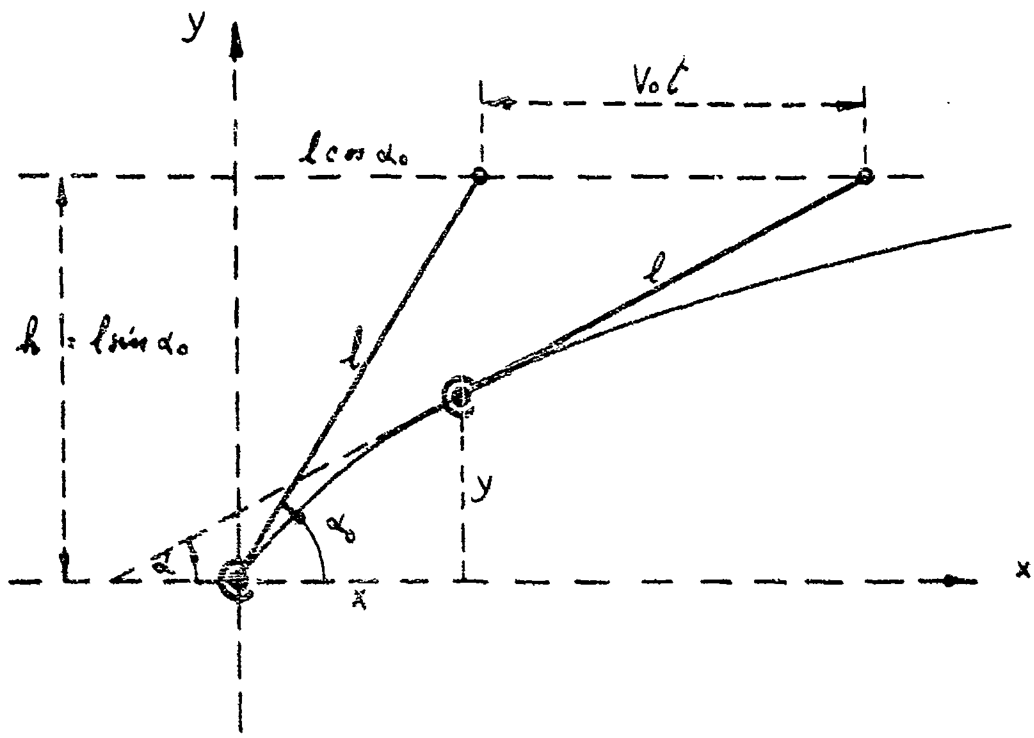


FIGURE 70

where V_0 is the landing velocity of the airplane. From these equations we obtain the result that the motion of the hook point is given by the equations:

$$x = V_0 t + l \cos\alpha_0 - l \, \mathrm{tgh} \frac{V_0}{l}(t - t_0) \tag{241}$$

$$y = h - \frac{l}{\cosh \frac{V_0}{l}(t - t_0)} \tag{242}$$

where the constant t_0 is determined by

$$\frac{V_0}{l} t_0 = \ln \mathrm{tg} \frac{\alpha_0}{2} . \tag{243}$$

This result can be written in dimensionless form. We introduce two constants x_0, y_0 setting

$$\frac{x_0}{l} = \cos\alpha_0 + \frac{V_0}{l} t_0 , \qquad \frac{y_0}{l} = \sin\alpha_0 \tag{244}$$

and a transformed time

$$\tau = \frac{V_0}{l}(t - t_0) \tag{245}$$

and obtain

$$\frac{x - x_0}{l} = \tau - \mathrm{tgh}\,\tau \tag{246}$$

$$\frac{y - y_0}{l} = - \frac{1}{\cosh \tau} \tag{247}$$

Figure 71 shows this curve.

The velocity components $\dot{x}$, $\dot{y}$ of the hook point are given by

$$\frac{\dot{x}}{V_0} = \mathrm{tgh}^2 \tau , \qquad \frac{\dot{y}}{V_0} = \frac{\mathrm{tgh}^2 \tau}{\sinh \tau} \tag{248}$$

and its velocity V by

$$\frac{V}{V_0} = \mathrm{tgh}\,\tau . \tag{249}$$

The largest upward velocity component $\dot{y}_{max} = \frac{V_0}{2}$.

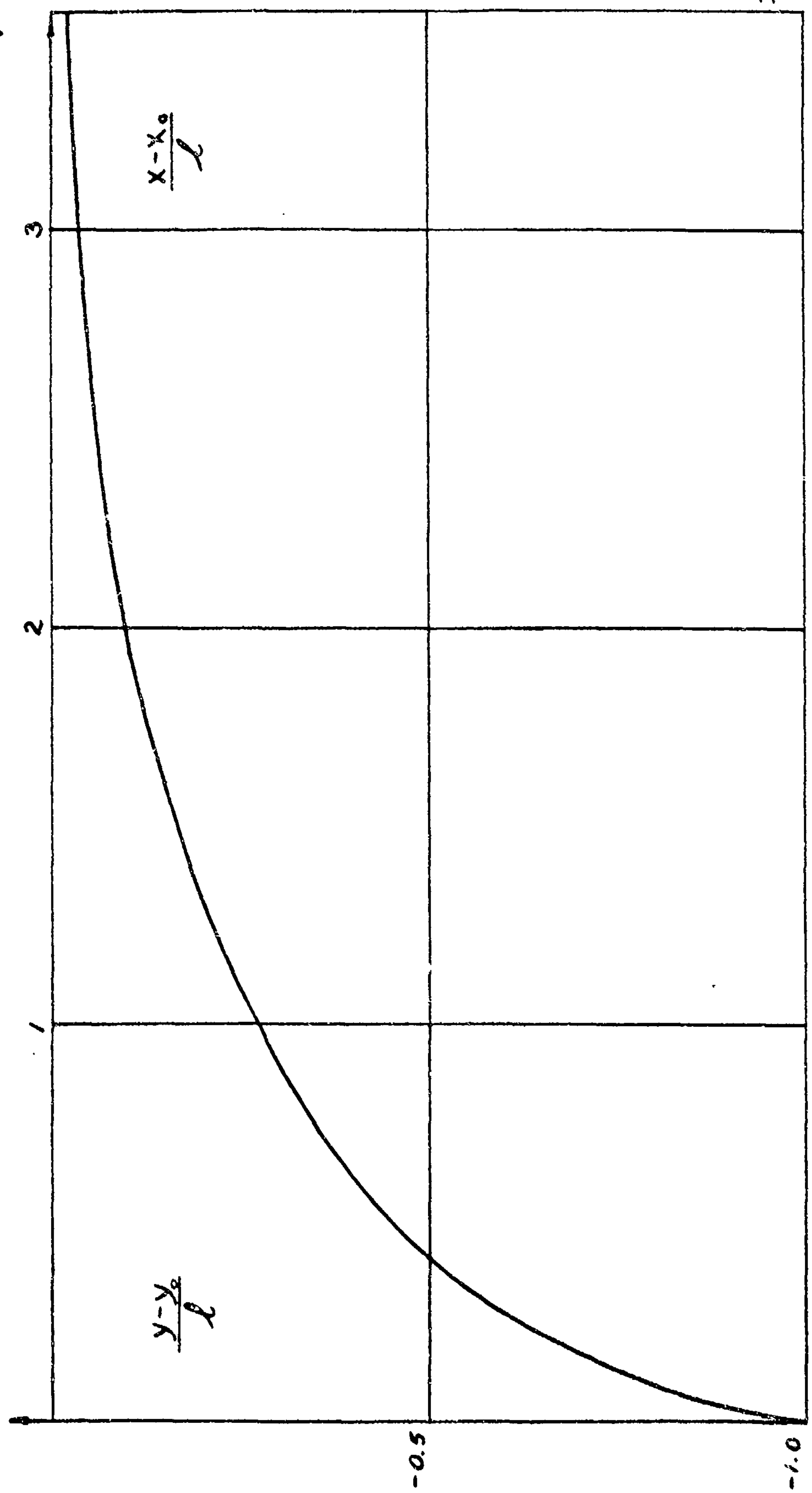


FIGURE 71

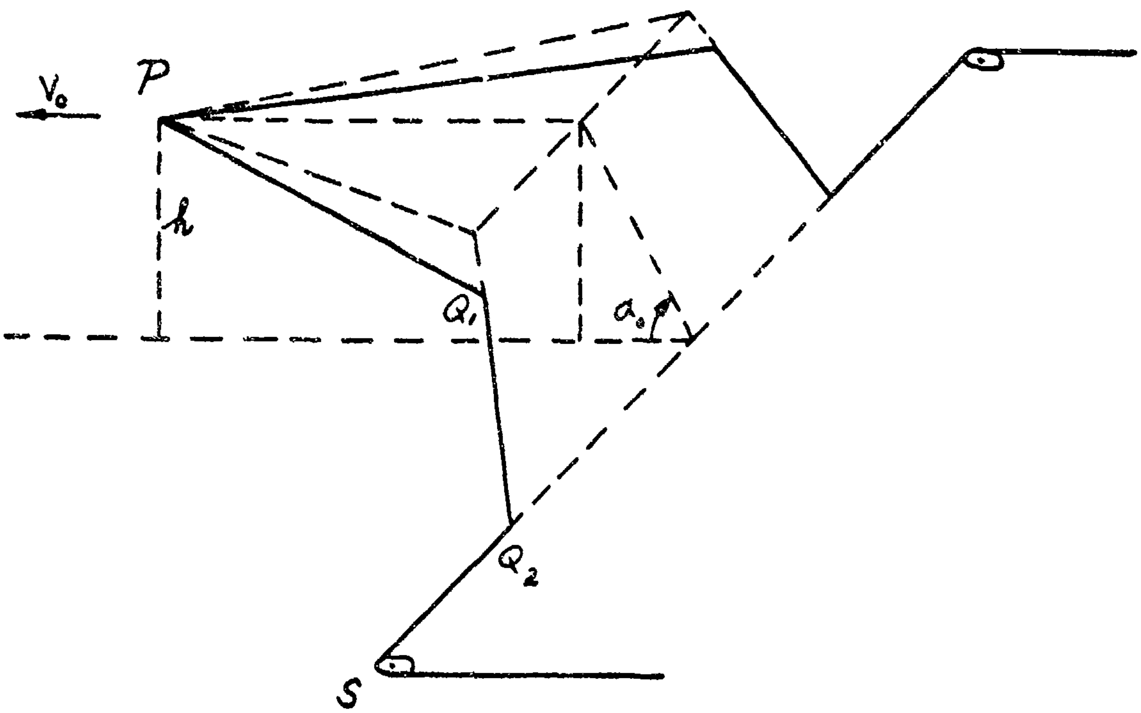


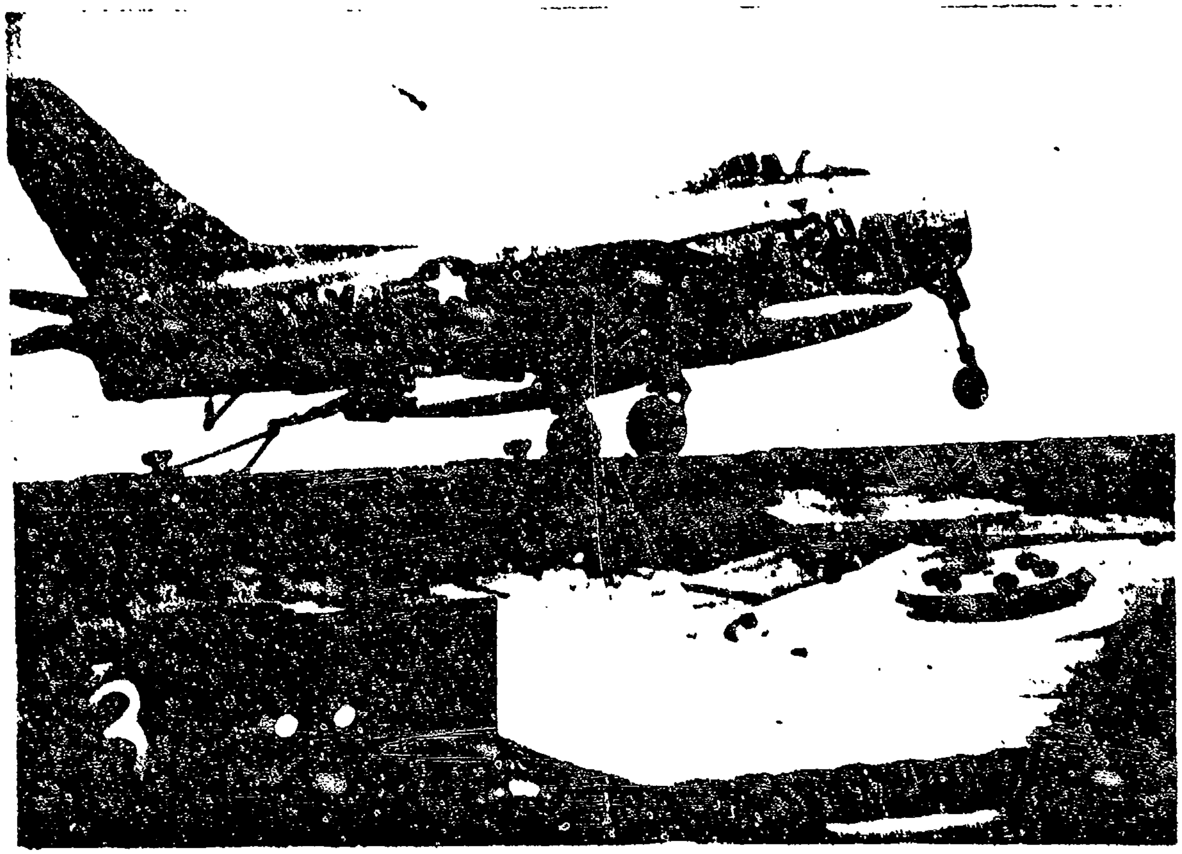
FIGURE 72

The hook point motion as described before is determined from kinematical considerations only. If we replace this motion approximately by two impact motions, one with the perpendicular impact direction α_0 and another oblique impact motion which results in a horizontal motion we obtain a cable shape as indicated in Figure 72. The cable tensions can be computed approximately from results obtained in Chapter IV.

We now have between the hook point P and one of the deck sheaves S' two kinks Q_1 and Q_2 in the cable. Both are moving toward S' with a speed which is determined by the stress in the cable. When Q_1 finally has reached the sheave S' the downward motion of the cable segment PQ_1 is stopped at S' and the cable will start to lie down on the deck a process which propagates from S' toward the hook point P with the effect that finally the hook point is pulled down to the deck. Immediately afterwards the hook point will start to move upward again similarly to the upward motion after the initial engagement.

The actual motion of the cable after the initial engagement is shown in the photographs, Figures 73 through Figure 81. The first of these pictures is remarkable as a document for the precision with which the initial kink wave occurs. In the following pictures it can be seen clearly how the initial cable triangle is bent over into the horizontal plane as described before. The last four pictures show on the other side of the deck how the kink Q_1 moves toward the sheave bringing the cable gradually down. In the last picture Q_1 has just reached the deck link. Shortly afterward, the cable will lie down on the deck near the sheave.

A serious problem for a safe operation of an arresting gear is presented by the downward motion of the deck link in its conventional form (see Figure 82). If there is a downward motion of the airplane tail after the engagement



(1) H/F
(1) 4/5
112-91

FIGURE 73
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299610(L)-1-57

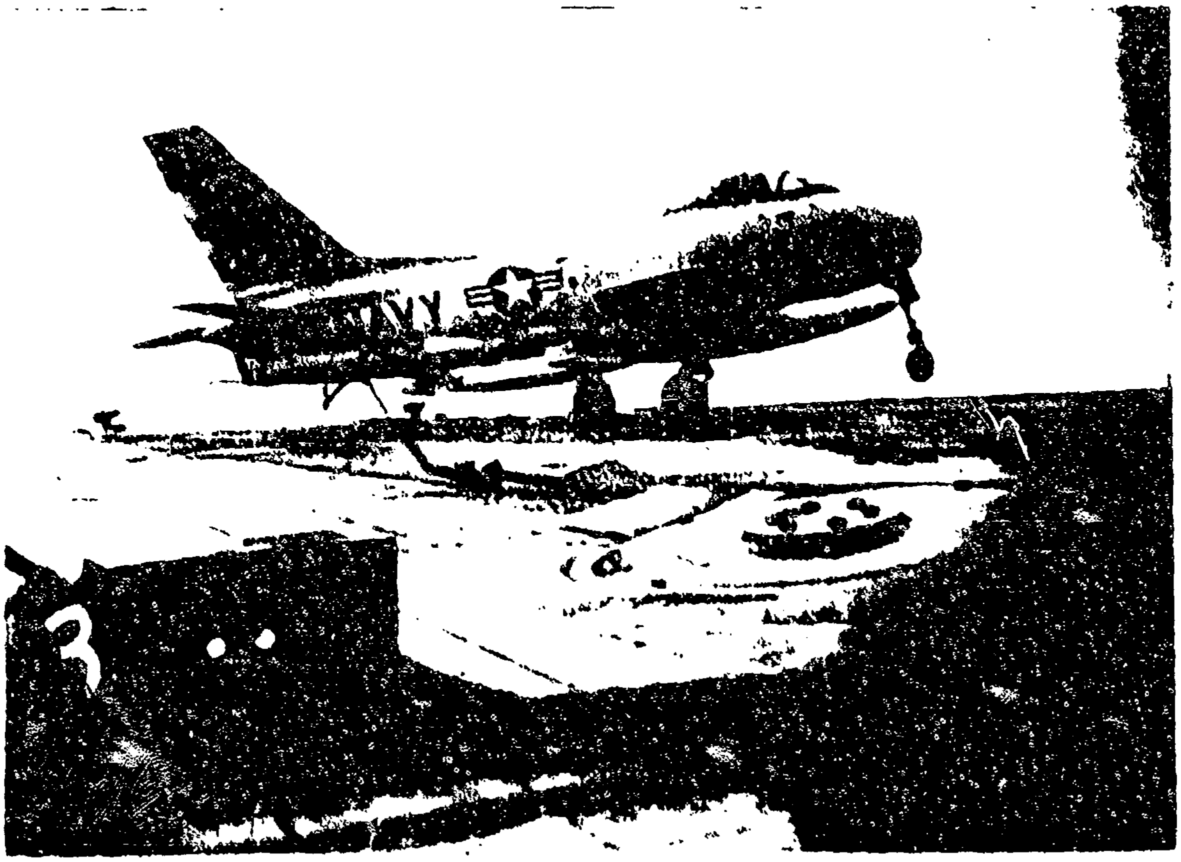
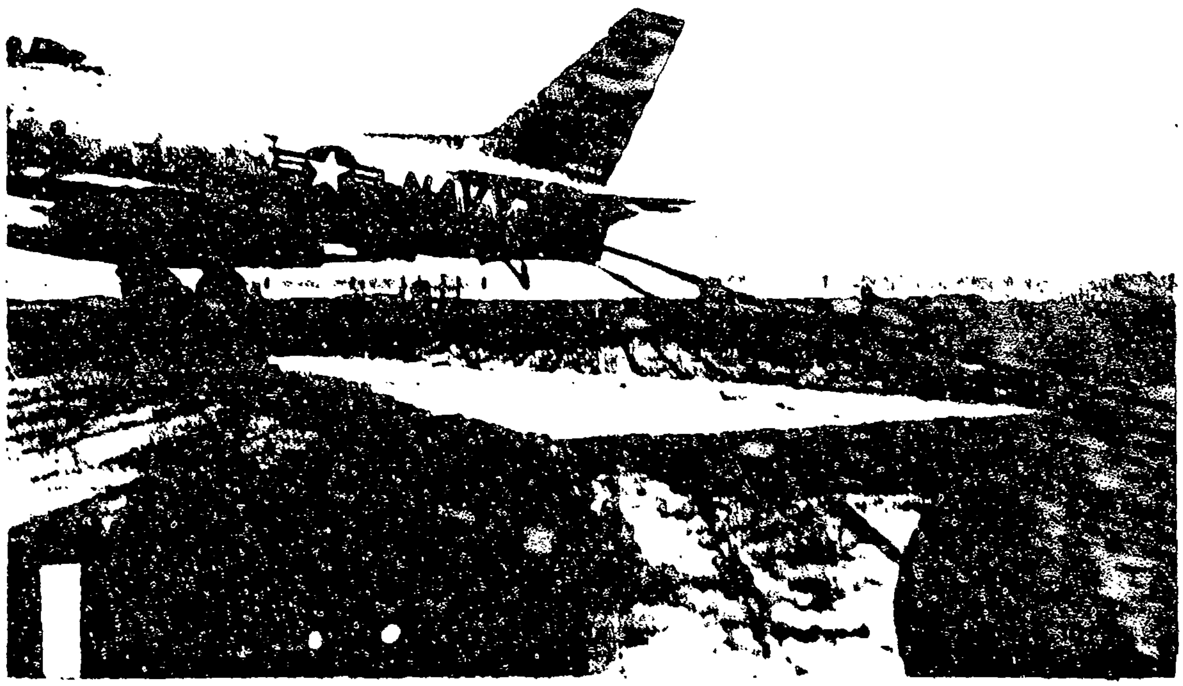


FIGURE 74
CABLE CONFIGURATION AFTER IMPACT

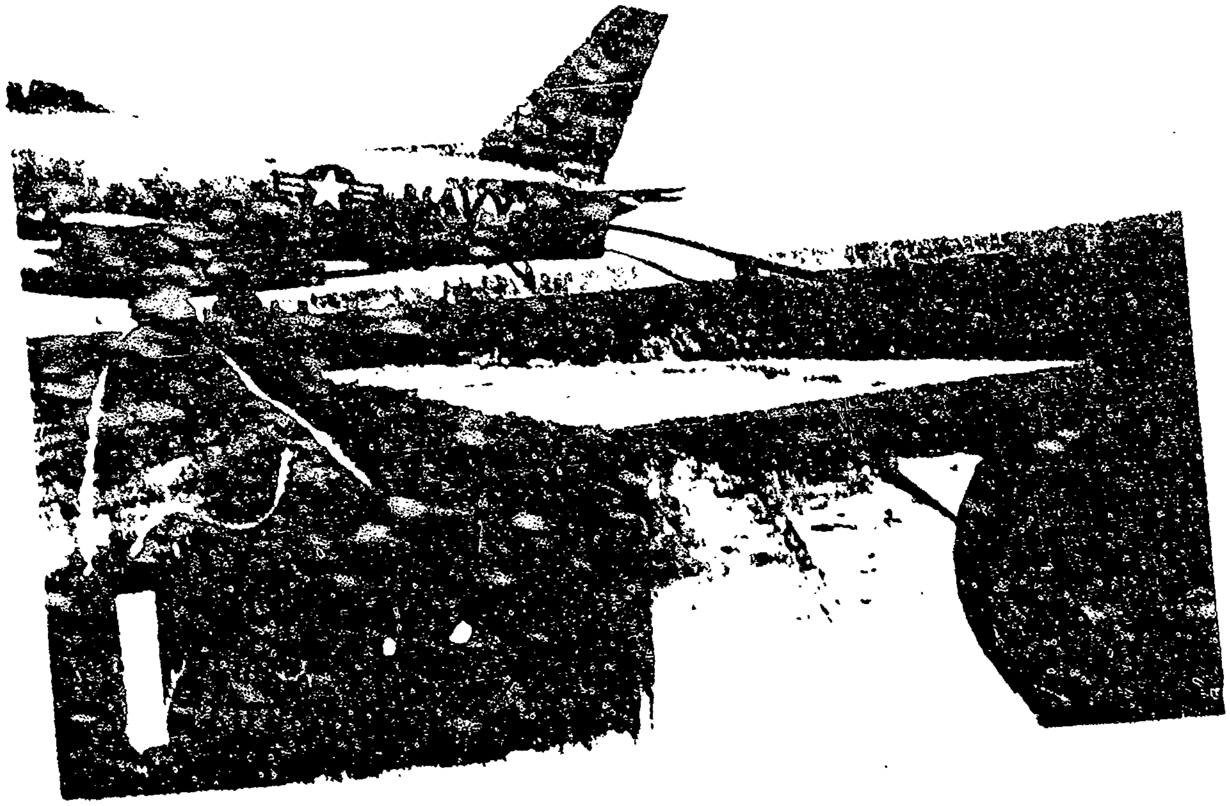
PHOTO NO: CAN-29961)-1-57



(1) H/T
(1) 4/S
112291

FIGURE 75
CABLE CONFIGURATION AFTER IMPACT

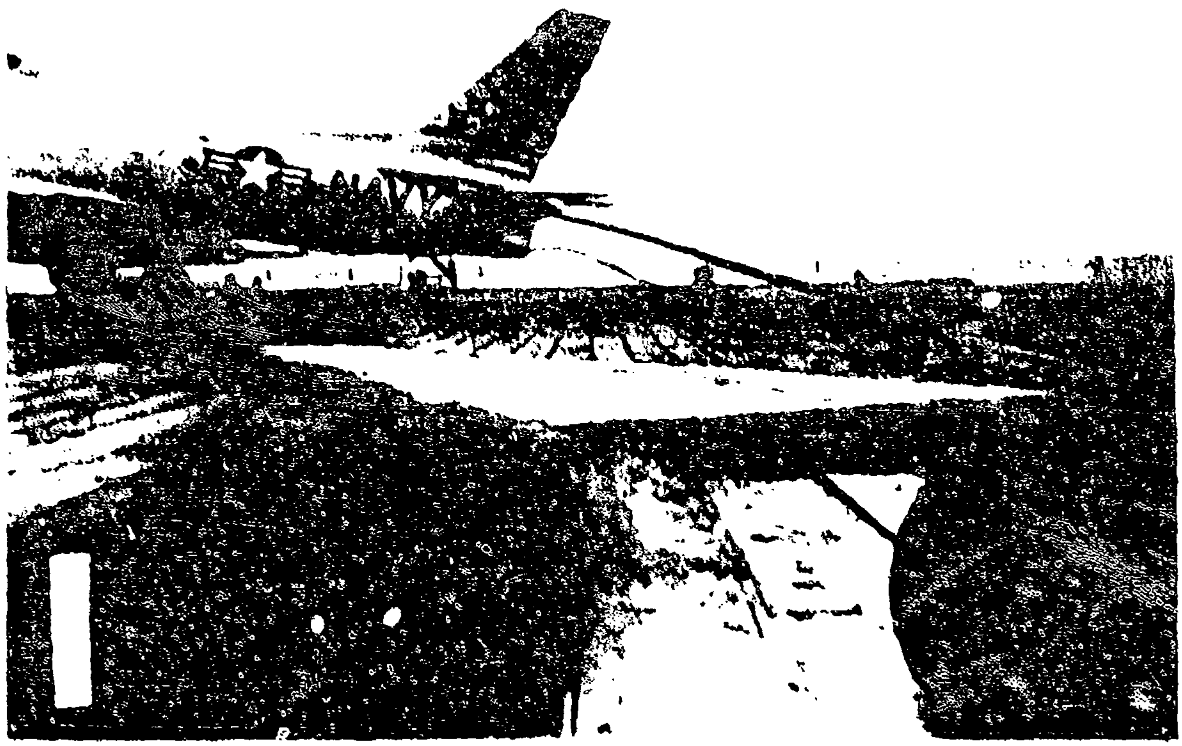
PHOTO NO: CAN-299614(L)-1-57



1. H/T
112291

FIGURE 76
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299616(L)-1-57



(1) H/T
(1) [illegible]
[illegible]

FIGURE 77
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299617(L)-1-57

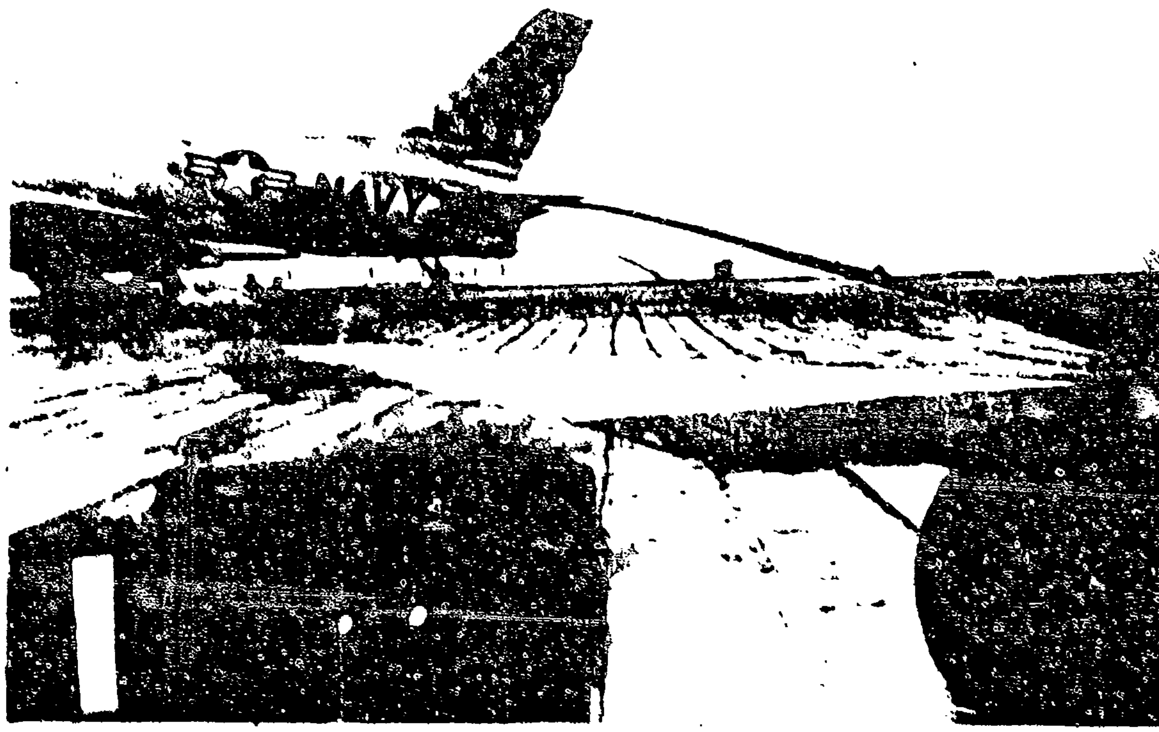


FIGURE 78
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299612(' 1-57

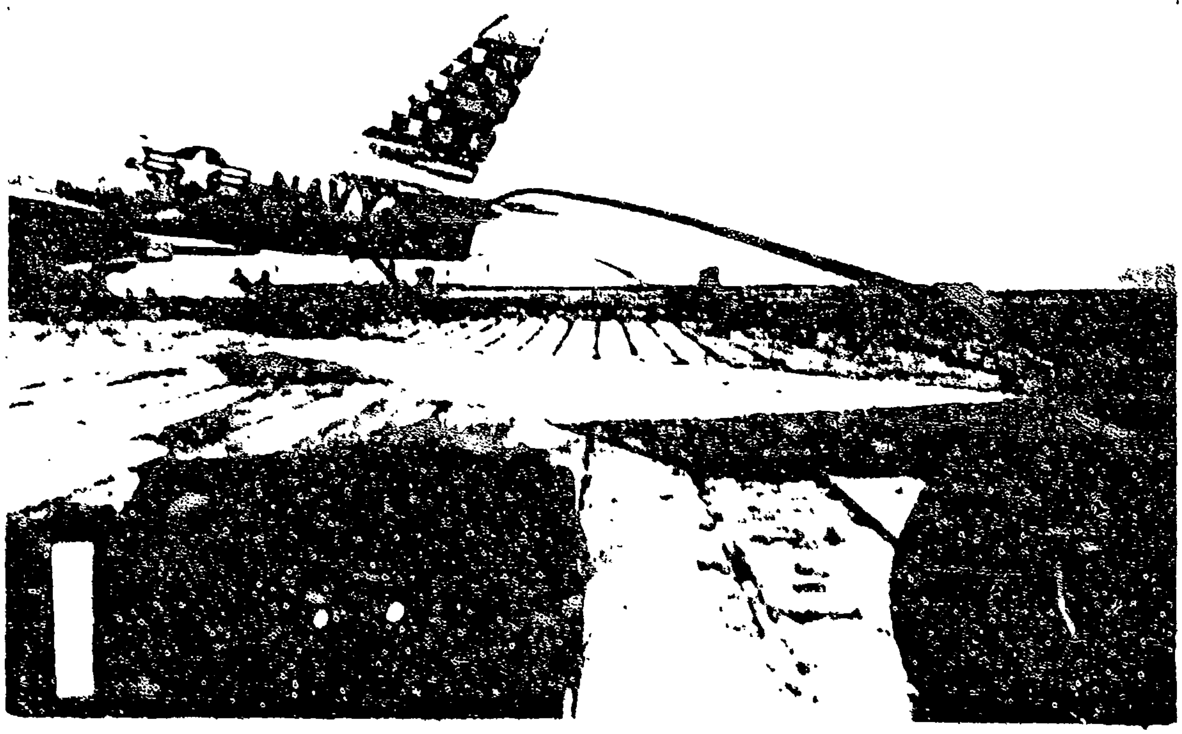
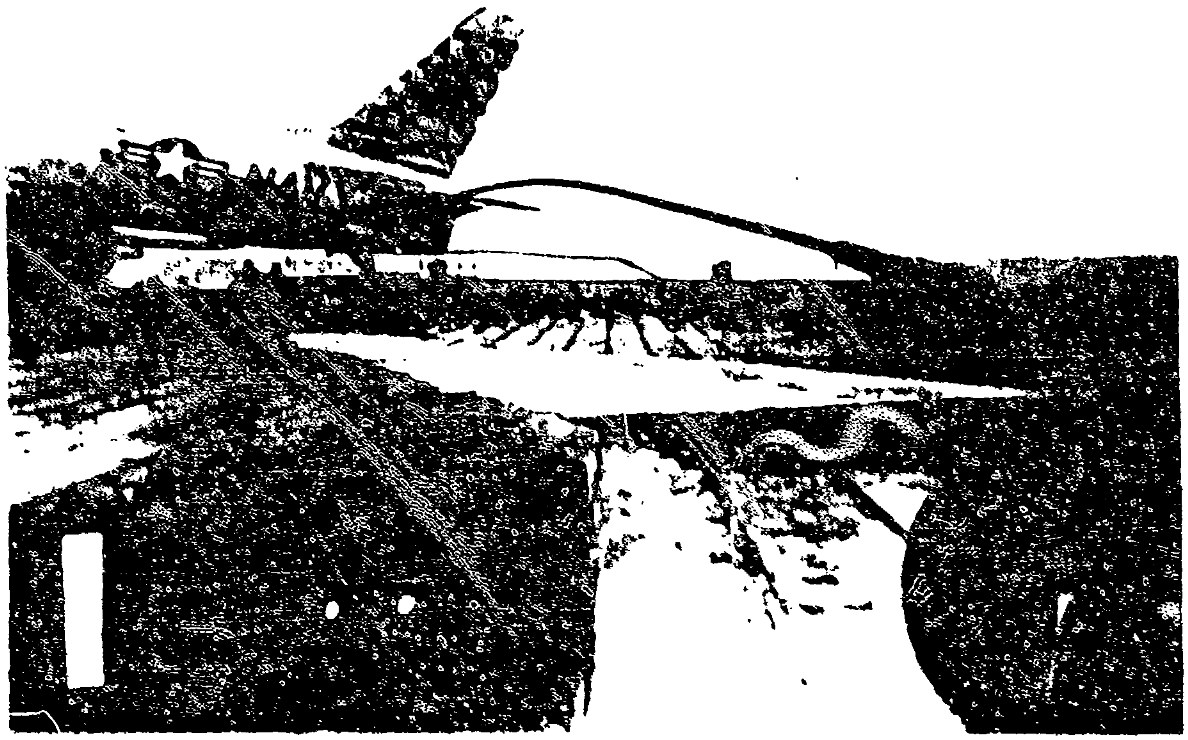


FIGURE 79
CABLE CONFIGURATION AFTER IMPACT

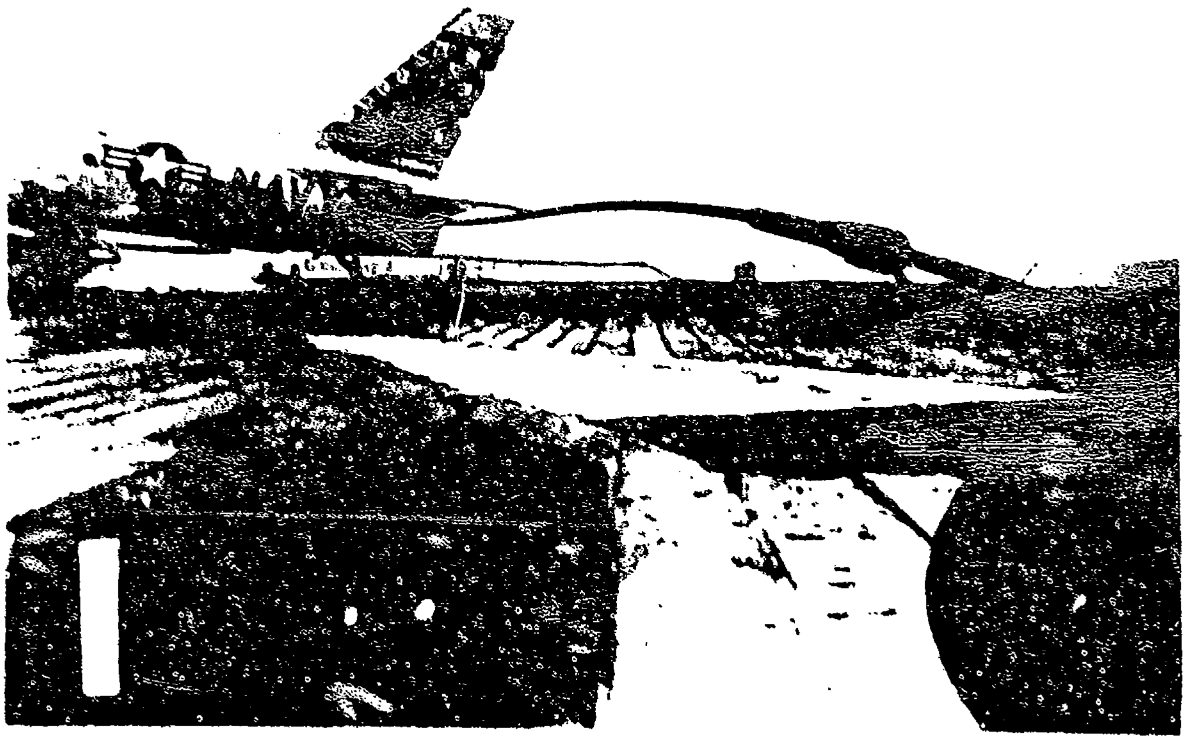
PHOTO NO: CAN-299618(L)-1-57



(1) H/F
(1) 43
11291

FIGURE 80
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299611(L)-1-57



(1) H/T
11229/

FIGURE 81
CABLE CONFIGURATION AFTER IMPACT

PHOTO NO: CAN-299615(L)-1-57

the downward motion of the link with the motion of the cable will be increased and in any case the link will hit the deck with a finite speed. By the deck link in this case a transverse cable impact stress is produced in that moment which propagates into the link while at the same time the terminal A (see Figure 82) is elastically repelled from the deck and produces a transverse impact at the terminal B. Superpositions of the produced stresses and their reflections combined with local stress concentrations due to the structure of the link can result in peak stresses which are sufficiently high in order to break the terminal B.

From the standpoint of cable dynamics this peak stress can be reduced mainly by reducing the angle α_o between hook and deck (see Figure 70) in the moment of engagement and by reducing large tension vibrations in the cable in order to avoid high tension in the cable in the moment of downward motion of the link.

The last condition will result in the use of slackless reeving systems or multi-cylinder engines. Then, however, deck links can be eliminated completely by shifting the cable about a small amount like the ribbon of a typewriter after each engagement. This can be done automatically with each retraction. Probability investigations have shown that a deck span length of 100 feet can be engaged more than 10 times as frequently as in the case of a deck pendant with links beside the fact that the total cable is usable for engagement.

Another problem related to the considerations of this section is that of double wire engagements. According to these considerations, the hook comes down to the deck after a certain time after the engagement which can be computed from the time a transverse wave needs to travel from the point of

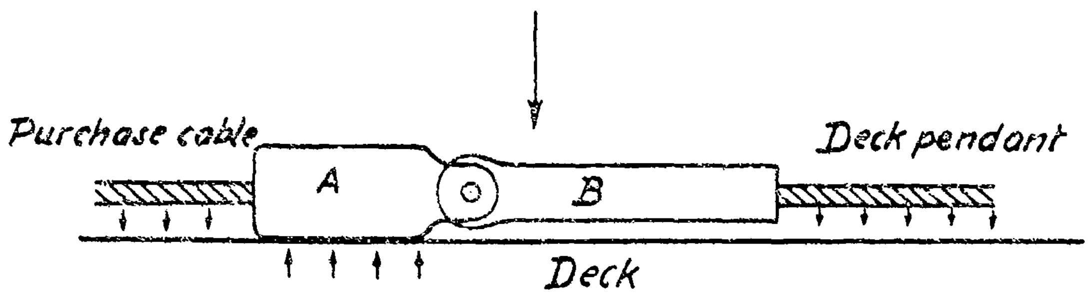


FIGURE 82

engagement in the center of the deck pendant up to the sheave and the time for such wave travel between sheave and hook. Both times are dependent on the stress in the cable and, therefore, on the engaging velocity. In Figure 83, these two times are denoted by t_1 and t_2. If the airplane passes over a deck pendant, the main wheels depress the cable. These transverse depressions travel to the deck sheave with a speed corresponding to the initial tension in the cable and return from the sheave with the same speed in form of an elevation. If this elevation arrives at the center of the deck pendant at the same time when the hook with the preceding deck pendant comes down to the deck, the possibility of a double wire engagement is created. Of course, the possibility of such coincidence depends on the spacings of the deck pendants and their lengths. beside the wave velocities. Figure 83 is computed for the case of a Mark 7 arresting gear with a 1-3/8" cable. The wave velocities have been computed from measured average cable tensions during the times in question. There are two sets of curves characterize by span lengths and by spacing distance. The intersection of two curves (one of each set) corresponds to a possible double wire engagement and determines the engaging velocity at which it can occur. The solid circular point, for instance, shows that at a deck span length of 93 feet and a spacing of 20 feet a double wire engagement is possible at an engaging velocity of about 73 knots and that the time $t_1 + t_2$ between first and second engagement is about .16 seconds. The plotted points show the existing conditions for several aircraft carrier classes which are favorable for double wire engagements. It can be shown, however, that double wire engagements are possible only for rather accurat center landings.

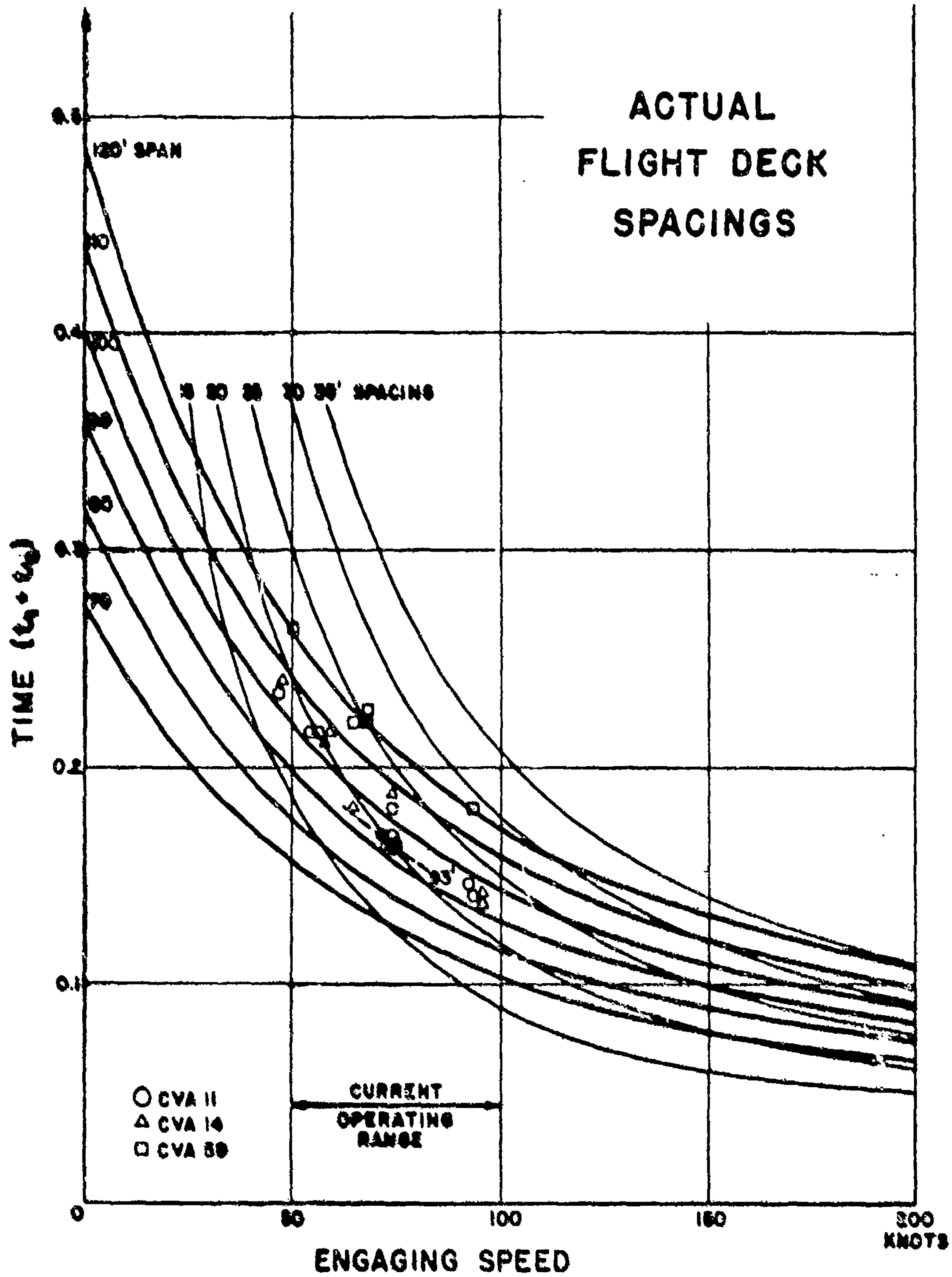

ACTUAL
FLIGHT DECK
SPACINGS
120' SPAN
110
100
90
80
70
15 20 25 30 35' SPACING
33'
0.5
0.4
0.3
0.2
0.1
0
50
100
150
200
KNOTS
TIME ($t_1 + t_2$)
ENGAGING SPEED
○ CVA 11
△ CVA 14
□ CVA 59
CURRENT
OPERATING
RANGE

FIG. 83

Final Remarks

This monography on cable dynamics is an extended version of a series of lectures the author presented in Philadelphia, Pa. and Los Angelos, Calif., in 1955 to selected groups of engineers and scientists of industry and Government in order to furnish information on problems and methods which the development of the aircraft arresting gear instigated. Though these investigations are of general technical and scientific interest, the purpose for which they have been done involves that this book is rather a presentation of selected subjects of cable dynamics than a textbook of the field. For those who wish to obtain a more complete knowledge of the total field of cable dynamics, some books which will close the existing gaps have been included in the list of references.

A considerable number of problems in this monography could be solved only by rather crude approximations. The possible solutions of others have been sketched only. Many analytical and experimental investigations are still underway. Therefore, in order to avoid a further delay of this publication, it is intended to add currently the results of such investigations to this monography in the form of appendices as soon as they are available.

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